Lesson 15: Rational Functions I

Def: A rational function is of the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. → have integer powers among the terms

Ex 1: Which of the following are rational function

(a) $f(x) = \frac{2+x}{3-x+x^2}$

Yes is a rational function

(b) $g(x) = \frac{1x^0}{x+2}$

Yes is a rational function

(c) $h(x) = \frac{\sqrt{x}+1}{x+2}$ → $x^{1/2}$

No b/c $\frac{1}{2}$ isn't an integer so $\sqrt{x}+1$ is not polynomial

(d) $k(x) = \frac{4x^5+3x}{1}$

Note: A polynomial is always a rational b/c I can always divide by 1.

Traits of rational functions

1. Intercepts
   - ↳ y-intercept → Plug in $x=0$
   - ↳ x-intercept → Plug in $y=0$ and solve
2. Discontinuities
   - ↳ Vertical Asymptotes (poles) [L15]
   - ↳ Holes [L16]
3. Other Asymptotes
   - ↳ Horizontal asymptotes [L15]
   - ↳ Slant Asymptotes [L16]
4. Multiplicity (after simplifying) } L17
   - ↳ Numerator: Behavior @ zeros

④ Multiplicity (after simplifying)
↳ Numerator: Behavior @ zeros
↳ Denominator: Behavior @ poles (VA)

L17

Horizontal & Vertical Asymptotes

An asymptote is a line (drawn as a dotted line) in the xy-plane that the function approaches.
↳ The graph of a function cannot cross a vertical asymptotes
↳ The graph of a function may touch/cross a horizontal asymptote.

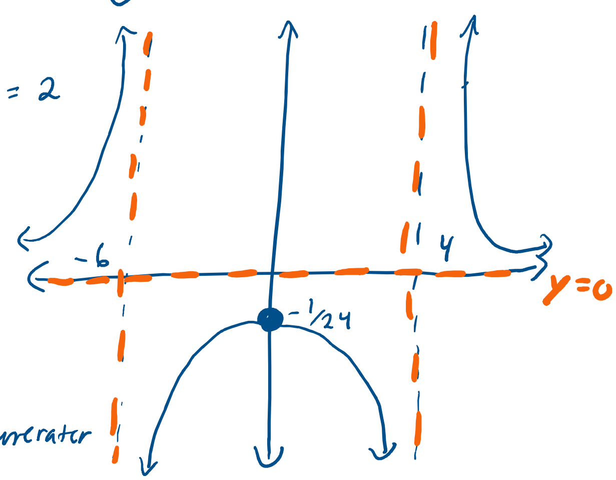
Ex 2: Graph the graph of f(x) and determine VA, and HA.

ⓐ $f(x) = \dfrac{1}{x^2+2x-24}$

VA: Set the denominator = 0 after simplifying the function.

$$x^2+2x-24=0$$
$$x^2+6x-4x-24=0$$
$$x(x+6)-4(x+6)=0$$
$$(x-4)(x+6)=0$$
$$x=4,-6$$

-24
$+6 \quad -4 = 2$



HA: So if the leading term degree of numerator less than the leading term degree of denominator, then HA is $y=0$.

y-intercept: Let $x=0$. $f(x) = \dfrac{1}{x^2+2x-24}$ $f(0) = \dfrac{1}{-24}$

$f(4.01) = \dfrac{1}{(10.01)(0.01)} = +$

$$f(4.01) = \frac{1}{(10.01)(0.01)} = +$$

ⓑ $g(x) = \dfrac{-x^2}{x^2+2x+1} = \dfrac{-x^2}{(x+1)^2}$

VA: Set the denominator $=0$ after simplifying the fraction.

$$(x+1)^2 = 0$$
$$x+1 = 0$$
$$x = -1$$

HA: If the leading term degree of numerator = leading term degree of denominator then it is the division of their coefficient, resp.

$$\frac{-\cancel{x^2}}{\cancel{x^2}} \to \frac{-1}{1} = -1 = y$$

ⓒ $h(x) = \dfrac{x}{x^2+4}$

VA: Set denominator $=0$ iafter simplifying

$$\left.\begin{aligned} x^2+4 &= 0 \\ x^2 &= -4 \\ x &= \pm\sqrt{-4} \end{aligned}\right\} \Rightarrow \text{No VA.}$$

VA: NONE

HA: Like (a) $y=0$ b/c x is degree 1 and x^2 is degree 2

Vertical Asymptotes:

① Simplify the rational function as much as possible

② If $(x-k)$ is a factor of the denominator, then $x=k$ is a VA.

Horizontal Asymptotes

Case 1: If deg(numerator) < deg(denominator), $y=0$ is a HA.

Case 2: If deg(numerator) = deg(denominator), then

$$y = \frac{\text{coefficient of leading term in numerator}}{\text{coefficient of leading term in denominator}}$$

Case 3: If deg(numerator) > deg(denominator), then no HA.

BUT there is another kind of asymptotes that happens, namely Slant asymptotes

Ex 3: Let $f(x) = \dfrac{3x}{x-20}$

(a) Find the domain of $f(x)$.

Denominator $\neq 0$

$x-20 \neq 0$

$x \neq 20 \Rightarrow (-\infty, 20) \cup (20, \infty)$

(b) Find the x-intercept(s) of $f(x)$.

Set $y=0$.

$$0 = \frac{3x}{x-20}$$

$$0(x-20) = \frac{3x\cancel{(x-20)}}{\cancel{x-20}}$$

$$\frac{0}{3} = \frac{3x}{3}$$

$$0 = x$$

x-int is @ $(0,0)$

(c) Find the y-intercept of $f(x)$.

Set $x=0$. $f(0) = \dfrac{3(0)}{0-20} = \dfrac{0}{-20} = 0$ } y-int is @ $(0,0)$

ⓓ Find VA.

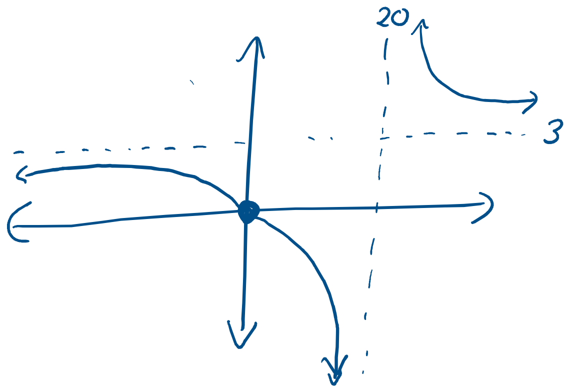
Well by part ⓐ, $x = 20$ is VA.

$$f(x) = \frac{3x^1}{x^1 - 20}$$

ⓔ Find HA.

$$y = \frac{3x}{x} = 3$$

ⓕ Graph



Ex 4: Let $g(x) = \dfrac{x+1}{x^2 - x - 6}$

ⓐ Find the domain of $g(x)$.

$$x^2 - x - 6 \neq 0$$
$$(x-3)(x+2) \neq 0$$
$$x \neq 3, -2 \quad \Rightarrow (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$$

ⓑ Find the x-intercept(s) of $g(x)$.

$$0 = \frac{x+1}{x^2 - x - 6}$$

$$0 = x + 1$$
$$x = -1 \quad \Rightarrow \text{x-int @ } (-1, 0)$$

ⓒ Find the y-intercept of $g(x)$.

$$g(0) = \frac{0+1}{0 - 0 - 6} = -\frac{1}{6} \Rightarrow \text{y-int @ } (0, -1/6)$$

ⓓ Find the VA.

By part (a), $x = 3$ and $x = -2$ are VAs.

ⓔ Find the horizonal asymptotes

ⓔ Find the horizonal asymptotes

deg (numerator) = 1
deg (denominator) = 2   so   deg(num) < deg(deno)   so $y=0$

Ex 5: Let $f(x) = \dfrac{5x^2+45x+70}{x^2-10x+21}$

ⓐ Find the domain of $f(x)$.

$x^2-10x+21=0$

$(x-7)(x-3)=0$

$x=3,7 \Rightarrow (-\infty,3)\cup(7,\infty)$

ⓑ Find the x-intercept(s) of $f(x)$.

$0=\dfrac{5x^2+45x+70}{x^2-10x+21}$

$5\cdot 70 = 350$

$35+10=45$

$0=5x^2+45x+70$

$0=5x^2+35x+10x+70$

$0=5x(x+7)+10(x+7)$

$0=(5x+10)(x+7)$