Monday, February 23, 2026 11:27 AM

Lesson 16: Rational Functions II

A rational function is of the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials.

Last time: VA & HA asymptotes
This time: Holes & Slant asymptotes

Holes

Ex 1: Let $f(x) = \dfrac{x+1}{x^2-x-2}$

ⓐ Find the domain of f.

$x^2 - x - 2 \neq 0$ b/c its a fraction

$(x-2)(x+1) \neq 0$

$x \neq 2, -1 \Rightarrow (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

ⓑ Find the simplified version of $f(x)$, if possible.

$$f(x) = \frac{x+1}{x^2-x-2} = \frac{\cancel{x+1}}{(x-2)\cancel{(x+1)}} = \frac{1}{x-2}$$

ⓒ Identify the VA and the holes

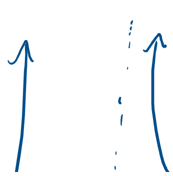
VA is the leftover in denominator.

VA: $x = 2$

Hole is when I cancel that factor.
Hole @ $x = -1$

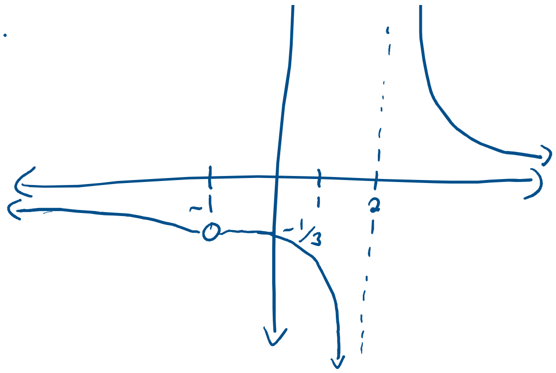
$f(x) = \dfrac{1}{x-2}$ $\Big\}$ Original function is $\frac{1}{x}$ but a right shift by 2

ⓓ Graph.



$f(-1) = \dfrac{1}{-1-2} = \dfrac{-1}{3} \Rightarrow$ Hole is at

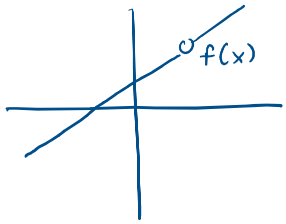
(d) Graph.



$f(-1) = \frac{1}{-1-2} = \frac{-1}{3} \Rightarrow$ Hole is at $(-1, \frac{-1}{3})$

__Types of discontinuities__

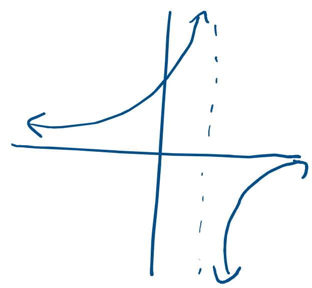
① Hole (removable discontinuity)



⇓

b/c that factor disappears

② VA: (infinite discontinuity)



⇓

b/c the functions y-values blow up to $+\infty$ or $-\infty$

__Ex 2:__ Find all holes and VA for the function

(a) $f(x) = \frac{x+1}{(x-2)(x+3)}$

Holes: Nothing b/c nothing cancels

VA: $x-2=0$, $x=2$ | $x+3=0$, $x=-3$

(b) $f(x) = \frac{\cancel{x-2}}{(\cancel{x-2})(x+3)}$

Hole: $x-2=0$, $x=2$ VA: $x+3=0$, $x=-3$

(c) $f(x) = \frac{(x-2)^{\cancel{2}}}{(\cancel{x-2})(x+3)}$

Hole: $x=2$ VA: $x=-3$

__Slant Asymptotes__

Recap HA: If deg(numerator) < deg(denominator), HA is $y=0$

If deg(numerator) = deg(denominator), HA is

Recap HA: If deg(numerator) < deg(denominator), HA is $y=0$

If deg(numerator) = deg(denominator), HA is

$$y = \frac{\text{numerator leading term}}{\text{denominator leading term}}$$

Note: this should be a #.

If deg(numerator) > deg(denominator), there is no HA.

BUT that means we have a slant asymptote.

A slant asymptote is a line in the xy-plane that the graph of a function approaches.

To find it we use long division or synthetic division.

Ex 3: Find the slant asymptote of $f(x) = \dfrac{4x^2+7x-15}{\underbrace{x+4}}$

long division $x+4 \overline{)\,4x^2+7x-15}$

Synthentic (Key is having a root)

$$\begin{array}{r|rrr} -4 & 4 & 7 & -15 \\ & \downarrow & -16 & 36 \\ \hline & 4 & -9 & 21 \\ & \downarrow & \downarrow & \downarrow \\ & x & c & \text{remainder} \end{array}$$

$f(x) = \boxed{4x-9} + \dfrac{21}{x+4}$

Slant!

Slant Asymptote @ $y = 4x-9$

Ex 4: Find the hole~~(s)~~, ~~VA(s)~~, HA and/or slant of the function

$$f(x) = \frac{8x^3+8x^2-8x-8}{x^2-x-2}$$

Simplify $f(x)$.

$$f(x) = \frac{8(x-1)(x+1)^{\cancel{2}}}{(x-2)\cancel{(x+1)}}$$

Numerator: $8x^3+8x^2-8x-8$

$= 8x^2(x+1) - 8(x+1)$

$= (8x^2-8)(x+1)$

$$f(x) = \frac{8(x\ \ldots)(\ldots)}{(x-2)\cancel{(x+1)}}$$

$$=(8x^2-8)(x+1)$$
$$=8(x^2-1)(x+1)$$
$$=8(x-1)(x+1)(x+1)$$

Hole @ $x=-1$

VA : $x=2$

$$f(x) = \frac{8(x-1)(x+1)}{x-2} = \frac{8(x^2-1)}{x-2} = \frac{8x^2-8}{x-2} = \frac{8x^2+0x-8}{x-2}$$

$$\begin{array}{r|rrr} 2 & 8 & 0 & -8 \\ & \downarrow & 16 & 32 \\ \hline & 8 & 16 & 24 \end{array}$$

Slant Asymptotes : $y = 8x+16$

$$f(x) = 8x+16 + \frac{24}{x-2}$$

Ex 5: Find a function for a rational function w/ the following properties:

① VA: $x=-5$, $x=-1$, $x=0$ → Deno. $(x+5)(x+1)x$ ✓

② HA @ $y=0$ → deg(numerator) = deg(denominator) ✗

③ Double zero at $x=4$ → Numerator $(x-4)^2$ ✓

$$\cancel{f(x)} = \frac{(x-4)^2}{(x+5)(x+1)x}$$

I need another factor on the top. It can't be $x+5$, or $x+1$, or x. b/c they would cancel and gives us a hole.

Let $f(x) = \dfrac{(x-4)^2(x+2)}{(x+5)(x+1)x}$ ✓

Ex 6: Find a function with the following properties

① Zeroes at $x=5$ and $x=6$ → Numerator $(x-5)(x-6)$ ✓

② HA @ $y=2$ → deg(num) = deg(deno) ✓ I need the leading term of the numerator to be 2.

③ VA @ $x=4$ → Deno. $(x-4)$ ✓

④ Hole @ $x=10$ → Top & Bottom $(x-10)$ ✓

term of the numerator
to be 2.

③ VA @ $x = 4$ → Deno. $(x-4)$ ✓

④ Hole @ $x = 10$ → Top & Bottom $(x-10)$ ✓

$$f(x) = \frac{2(x-5)(x-6)(x-10)}{(x-4)^2(x-10)}$$