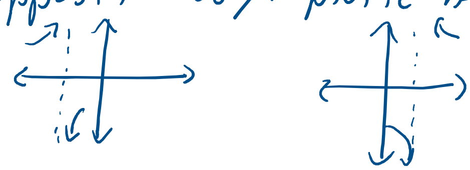
Reminders: Written HW 5 Due Friday
Exam 2 next Wednesday @ 8pm
in ES2107

Lesson 17: Graphs of Rational Functions

- For rational functions, there is multiplicity in the numerator and denominator.
  - ↳ Numerator
    - ↳ Zero with odd multiplicity: cross x-axis and opposite signs (+/− or −/+)
    - ↳ Zero with even multiplicity: touch x-axis and same signs (+/+ or −/−)
  - ↳ Denominator
    - ↳ Zero with odd multiplicity: opposite asymptotic behavior



    - ↳ zero with even multiplicity: Same asymptotic behavior

☆ Before analyzing the multiplicity, make sure rational function is simplified.

- $R(x) = \frac{P(x)}{Q(x)}$ Factor $P(x)$ and $Q(x)$ and cancel like terms.
  - ↳ Domain is zeros of $Q(x)$ before canceling

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↳ y-intercept: Set $x=0$ to find y.

↳ x-intercept: Set $y=0$ to find x. [Trick is it is enough to set $P(x)=0$]

↳ Vertical Asymptotes (VA): zeros of $Q(x)$ after simplifying the fraction

↳ Holes : zeros of $Q(x)$ after canceling

↳ Horizontal Asymptotes & Slant Asymptotes

↳ $\deg(P) < \deg(Q)$ HA at $y=0$

↳ $\deg(P) = \deg(Q)$ HA at $y = \dfrac{(\text{leading term of } P)}{(\text{leading term of } Q)}$

↳ $\deg(P) = \deg(Q)+1$ HA doesn't exist

BUT slant asymptote

Find that by synthetic or long division

↳ $\deg(P) > \deg(Q)+1$ No HA or Slant Asymptotes

Ex 1: Sketch the graph of

$$f(x) = \frac{3x^2+6x-9}{x^2-6x+5}$$

Include all important characteristic.

Domain: $x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=1,5$

Domain is $(-\infty,1)\cup(1,5)\cup(5,\infty)$

Now simplify $f(x)$.

$3x^2+6x-9 = 3(x^2+2x-3)$

$= 3(x+3)(x-1)$

$$f(x) = \frac{3(x+3)\cancel{(x-1)}}{(x-5)\cancel{(x-1)}}$$

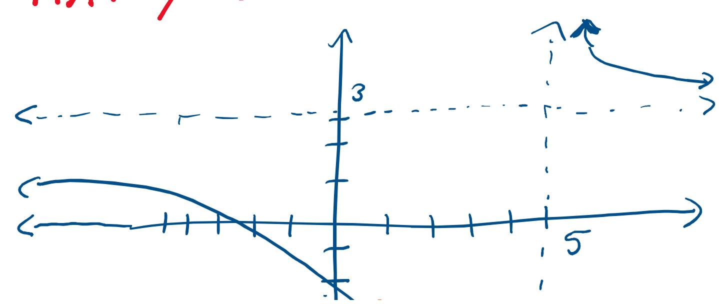
Hole: $x-1=0 \iff x=1$ $f(1) = \frac{3\cdot 4}{-4}$

$(1, -3)$

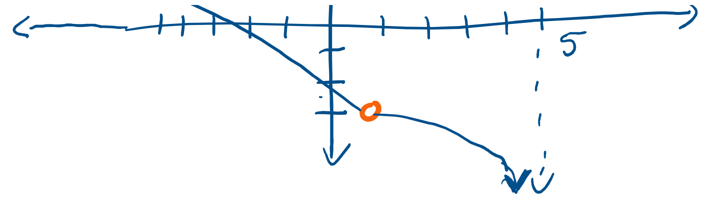
VA: $x-5=0 \iff x=5$

Now $f(x) = \dfrac{3(x+3)}{x-5} = \dfrac{3x+9}{x-5}$

HA: $y=3$



$$3x^2+6x-9 = 3(x^2+2x-3)$$
$$= 3(x+3)(x-1)$$



$$f(x) = \frac{3x-9}{x-5}$$

$x-5$ is in denominator and
has odd multiplicity
$\Rightarrow$ opposite asymptotic behavior

Ex 2: Sketch the graph

$$f(x) = \frac{x^3+5x^2+3x-9}{x^2+6x+8}$$

Include important characteristic

Domain: $x^2+6x+8=0$
$(x+4)(x+2)=0$
$x=-2,-4$

Domain is $(-\infty,-4)\cup(-4,-2)\cup(-2,\infty)$

Simplify $f(x)$.

$x^3+5x^2+3x-9 \longrightarrow P(1) = 1+5+3-9=0$

this can't be factored by grouping

By Fundamental Thm of Alg,

roots = $\dfrac{\text{(factors of leading term coefficient in numerator)}}{\text{(factors of leading term coefficient in denominator)}} = \dfrac{\pm 1}{\pm 1} = 1, -1$

$(a-b)^2$
$=$
$a^2-2ab+b^2$

Synthentic:

$$\begin{array}{r|rrrr} 1 & 1 & 5 & 3 & -9 \\ & \downarrow & 1 & 6 & 9 \\ \hline & 1 & 6 & 9 & 0 \end{array}$$

$\Rightarrow x^3+5x^2+3x-9 = (x-1)(x^2+6x+9)$
$= (x-1)(x+3)^2$

So $f(x) = \dfrac{(x-1)(x+3)^2}{\quad} = \dfrac{x^3+5x^2+3x-9}{\quad}$

So $f(x) = \dfrac{(x-1)(x+3)^2}{(x+4)(x+2)} = \dfrac{x^3+5x^2+3x-9}{x^2+6x+8}$

Hole: None

VA: $x = -4, -2$

HA: None

Slant Asymptote: Yes. To find do long division. (Why not synthenic b/c x^2+6x+8 which isn't degree 1)

$$\begin{array}{r} x-1 \\ x^2+6x+8 \enclose{longdiv}{x^3+5x^2+3x-9} \\ -(x^3+6x^2+8x) \\ \hline -x^2-5x-9 \\ -(-x^2-6x-8) \\ \hline x-1 \end{array}$$

① $\dfrac{x^3}{x^2} = x$

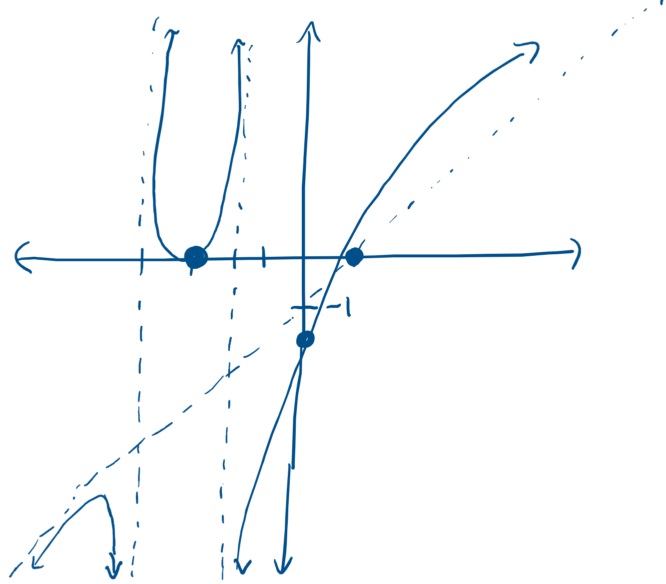
$\downarrow$

$x(x^2+6x+8) = x^3+6x^2+8x$

② $\dfrac{-x^2}{x^2} = -1$

$\downarrow$

$-1(x^2+6x+8) = -x^2-6x-8$



$f(x) = \dfrac{(x-1)(x+3)^2}{(x+4)(x+2)}$

y-intercept: $x=0$ $\quad f(0) = \dfrac{-1(3)^2}{4(2)} = \dfrac{-9}{8}$

$(0, -9/8)$

x-intercept: $y=0$ $\quad 0 = \dfrac{(x-1)(x+3)^2}{(x+4)(x+2)}$

$0 = (x-1)(x+3)^2$

$x = 1, -3$

$(1,0)$, $(-3,0)$

$\downarrow$ odd mult. $\downarrow$ cross

$\downarrow$ even mult. $\downarrow$ touche

$(x+4)(x+2)$

Each as odd multiplicity
opposite directions