Lesson 18 - Inverse Functions

One-to-one Functions

- For a relation to be a function, for each input there must be only one output
- For a function to be one-to-one, for each output there must be only one input

ex

$f(0)=1$

(a) $\{(0,1), (3,5), (2,6)\}$

input	output
0	1
2	5
3	6

$0 \to 1$, $2 \to 6$, $3 \to 5$

✓ one-to-one

(b) $\{(1,3), (2,7), (5,3)\}$

$1 \to 3$, $2 \to 7$, $5 \to 3$

✗ not one-to-one

(c) $h(x) = x^2$ on dom. $(-\infty, \infty)$

$(-2)^2 = 4$

$-2 \to 4$, $2 \to 4$

✗ not one-to-one

- recall that the vertical line test determines whether or not something is a function
- to determine if a function is one-to-one, we can use the horizontal line test

$y=k$ if there is horiz. line with >1 intersection $\Downarrow$ not one-to-one

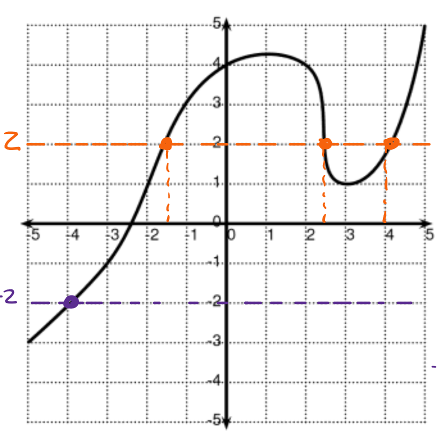
if there is no such horiz line (every horiz line has only 1 intersection) $\Downarrow$ is one-to-one

ex

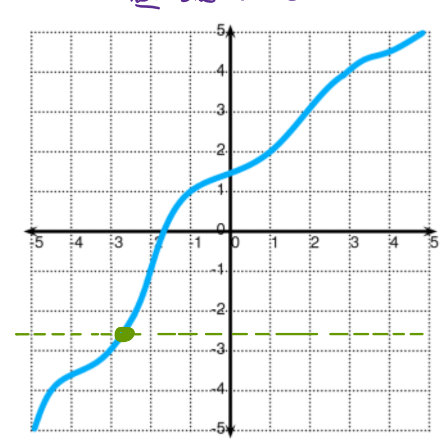
$y=k$

$y=2$

$y=-2$



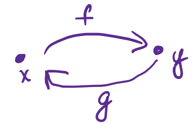
not one-to-one



one-to-one

Inverse Functions

- The function g is the inverse of f exactly when, for every x in the domain of f, $\boxed{\text{if } f(x)=y, \text{ then } g(y)=x.}$
  - ↳ f has an inverse if we can "undo" f, if f is reversible



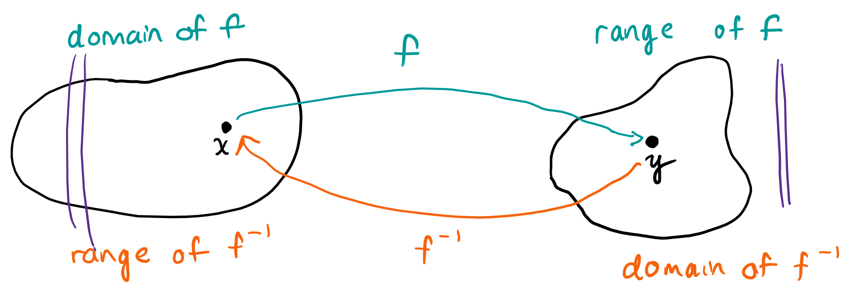
  - ↳ what does this say about . . .
    - $g(f(x))$? $\quad y=f(x) \leftrightarrow g(y)=x$

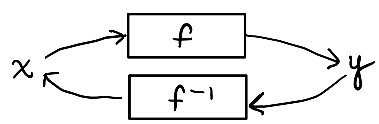
      $g(y) = x$
    - $f(g(y))$?

      $f(x) = y$

  if $f(g(x))=x$ & $g(f(x))=x$ then g is the inverse of f

↳ if g satisfies this, we write $g = f^{-1}$ ("f inverse")





$\sin^{-1} \neq \frac{1}{\sin}$

⚠ $f^{-1}(x) \neq (f(x))^{-1}$ ------→ $(f(x))^{-1} = \frac{1}{f(x)}$ ⚠

f^{-1} ↑ inverse function; $(f(x))^{-1}$ ↑ reciprocal ← $(4)^{-1} = \frac{1}{4}$

ex Domain: all Purdue students

(a) The function $B(s)$ takes a Purdue student and outputs their birthday.
If $B(s_0) = 08/03/2006$, can we say for sure who s_0 is? → no, multiple students have same b-day
Does B have an inverse?
↓
no

student → birthday
not reversible

B not one-to-one

(b) The function $P(s)$ takes a Purdue student and outputs their PUID.
If $P(s_0) = 001\,234\,5678$, can we say for sure who s_0 is? → yes
each student has a unique PUID
Does P have an inverse?
↓
yes

student ⇄ PUID
reversible

P is one-to-one

- f has an inverse $\leftrightarrow$ f is one-to-one $\leftrightarrow$ the graph of f passes the horizontal line test ex Let f be a one-to-one function with inverse f^{-1}.

$f(x)=y \iff f^{-1}(y)=x$

(a) If $f(1)=3$, find $f^{-1}(3)$.

$x=1 \quad y=3$ $\qquad f^{-1}(3)=1$

$(1,3)$ on graph of f $\qquad (3,1)$ is on graph of f^{-1}

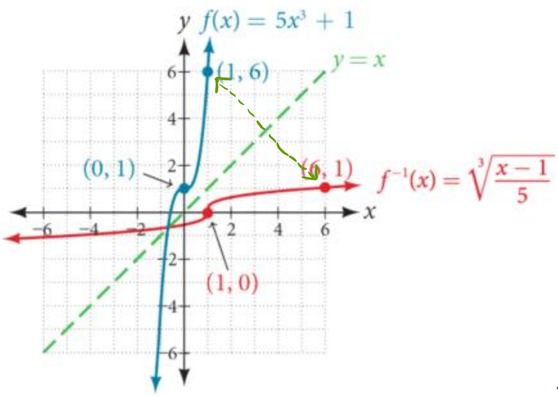
(b) If $f^{-1}(10)=13$, find $f(13)=10$

$y=10 \quad x=13$

$(10,13) \qquad (13,10)$

- The graph of f^{-1} is the graph of f mirrored about $y=x$:

reflecting over $y=x \iff$ switching x & y



Check: are these really inverses?

verify:

1) $f(f^{-1}(x))=x$

$f\left(\sqrt[3]{\frac{x-1}{5}}\right)=5\left(\sqrt[3]{\frac{x-1}{5}}\right)^3+1$

$=5\left(\frac{x-1}{5}\right)+1$

$=(x-1)+1 \quad =x$ ✓

2) $f^{-1}(f(x))=x$

$f(x) = 5x^3+1$

$f^{-1}(5x^3+1)=\sqrt[3]{\frac{(5x^3+1)-1}{5}}=\sqrt[3]{\frac{5x^3}{5}}$

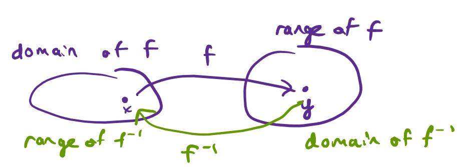
$=\sqrt[3]{x^3}=x$ ✓

Finding Inverses

start write $y=f(x)$

- how to find inverses? If the inverse of f exists, ...
  1) Set $x=f(y)$ (switch x & y)
  2) solve for y
  3) Set $f^{-1}(x)=y$

Example For each function, find its inverse, the domain of its inverse, and the range of its inverse.



(a) $f(x)=2x+3$

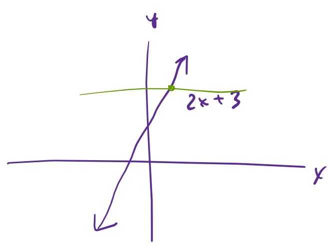
$y=2x+3$

1) $x=f(y)=2y+3$

2) $x-3=2y$

$y=\frac{x-3}{2}$

3) $f^{-1}(x)=\frac{x-3}{2}=\frac{1}{2}x-\frac{3}{2}$



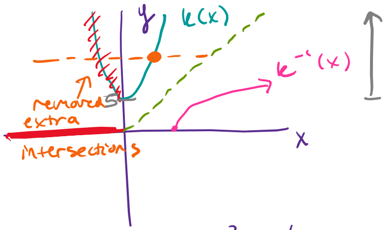
domain of f: $(-\infty,\infty) \iff$ range of f^{-1}

range of f: $(-\infty,\infty) \iff$ domain of f^{-1}: $(-\infty,\infty)$

• Sometimes we restrict a function to a domain where it is one-to-one and then we can find an inverse (to the restricted function)

(b) $k(x) = 2x^2 + 5, \quad x \geq 0$

all (x,y) in the graph of $k(x)$ have $x \geq 0$, $y \geq 0$



$x = 2y^2 + 5$

$2y^2 = x - 5$

$y^2 = \dfrac{x-5}{2}$

domain of k: $[0, \infty)$ = range of k^{-1}

$a^2 = b$

$a = \pm\sqrt{b}$

$k^{-1}(x) = y = \sqrt{\dfrac{x-5}{2}}$

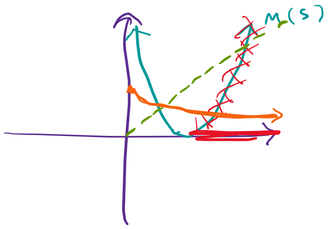
$\dfrac{x-5}{2} \geq 0$

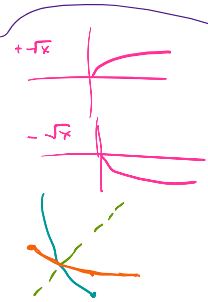
domain of k^{-1}: $[5, \infty)$ = range of k

$\dfrac{x-5}{2} \geq 0 \rightarrow x - 5 \geq 0 \rightarrow x \geq 5$

(c) $m(s) = (s-3)^2, \quad s \leq 3$

shift s^2 right 3





$y = (x-3)^2$

$x = (y-3)^2$

$y - 3 = -\sqrt{x}$

$y = -\sqrt{x} + 3$

$m^{-1}(s) = -\sqrt{s} + 3$

$s \leq 3$

$s - 3 \leq 0$

$m^{-1}(m(s)) = \underbrace{-\sqrt{(s-3)^2}}_{} + 3$

$= (s-3) + 3 = s$

ex $s = 2$

$s - 3 = -1$

$-\sqrt{(-1)^2} = -1$

$= -\sqrt{1}$

(d) $g(x) = \dfrac{x+2}{x-6}$

$x = \dfrac{y+2}{y-6}$

$x(y-6) = y+2$

$xy - 6x = y + 2$
$+6x \qquad +6x$

$xy = y + 2 + 6x$
$-y \qquad -y$

$xy - y = 6x + 2$

$y(x-1) = 6x + 2$

$y = \dfrac{6x+2}{x-1}$

$\boxed{g^{-1}(x) = \dfrac{6x+2}{x-1}}$

domain of $g = (-\infty, 6) \cup (6, \infty)$ = range of g^{-1}

domain of $g^{-1} = (-\infty, 1) \cup (1, \infty)$ = range of g

(e) $h(x) = -5x + 2$

$h^{-1}(x) = \dfrac{x-2}{-5}$

$x = h(y) = -5y + 2$

$x - 2 = -5y$

$y = \dfrac{x-2}{-5} = -\frac{1}{5}x + \frac{2}{5}$

1) Set $x = f(y)$
2) Solve for y
3) Set $f^{-1}(x) = y$