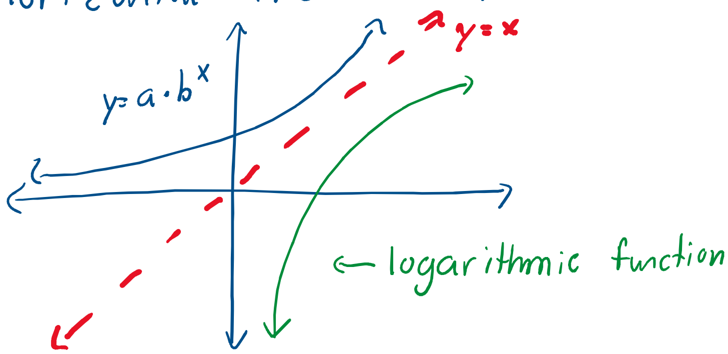
Lesson 20: Logarithmic Functions

Exponential Function is $f(x) = a \cdot b^x$ where $a \neq 0$, $b > 0$, $b \neq 1$

(b ↑ base)

Last lesson, we noted that exponential functions pass the horizontal line test $\Rightarrow$ one-to-one $\Longrightarrow$ inverse



Logarithmic function is the inverse function of an exponential function.

$$f(x) = y \iff f^{-1}(f(x)) = f^{-1}(y)$$
$$x = f^{-1}(y)$$

$$\log_b(x) = y \iff b^y = x$$

Special logarithms

① $\log(x) = \log_{10}(x)$ ② $\ln(x) = \log_e(x)$

Ex 1: Write the equation in exponential form

ⓐ $\log_2(x) = 12 \iff 2^{12} = x$

ⓑ $\log_3(x) = y \iff x = 3^y$

3 3

(b) $\log_3(x) = y \iff x = 3$

$3 \qquad 3$

(c) $\log(x) = 5$

$\log_{10}(x) = 5 \iff x = 10^5$

(d) $\ln(3) = y$

$\log_e(3) = y \iff 3 = e^y$

Ex 2: Write the equation in logarithmic form:

(a) $10 = 7^t$

$\log_7(10) = \log_7(7^t)$

$\log_7(10) = t$

(b) $x^{-2} = y$

$\log_{\cancel{x}}(\cancel{x}^{-2}) = \log_x(y)$

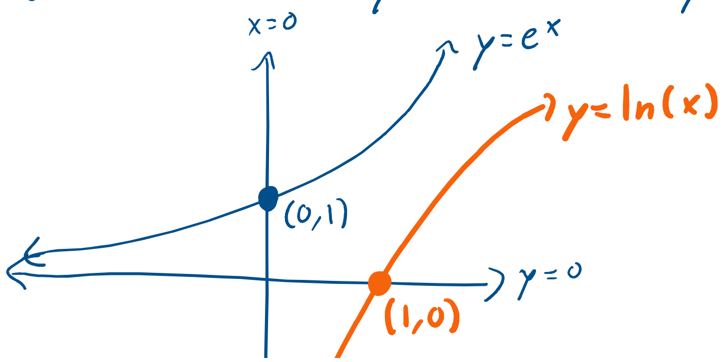
$-2 = \log_x(y)$

(c) $a^5 = 20$

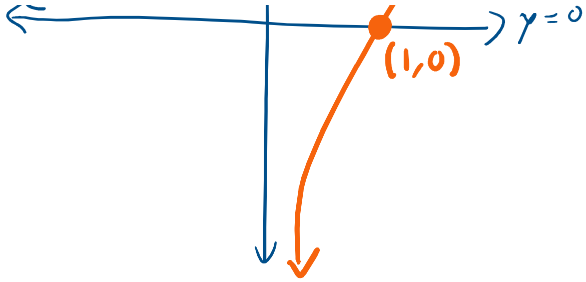
$\log_a(a^5) = \log_a(20)$

$5 = \log_a(20)$

Discussion on $y = \ln(x)$ and $y = e^x$ graphs



	$f(x) = e^x$	$g(x) = \ln(x)$
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$



Range	$(0,\infty)$	$(-\infty,\infty)$
x-intercept	None	$(1,0)$
y-intercept	$(0,1)$	None
Asymptote	HA @ $y=0$	VA @ $x=0$

Ex 3: Find the domain of the following functions

Domain of $\ln(?) \to ? > 0$

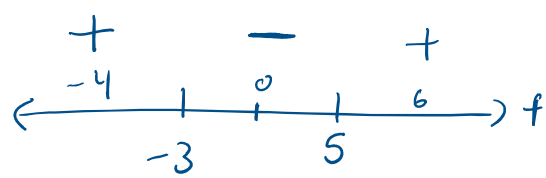
$\log(?) \to ? > 0$

$\log_a(?) \to ? > 0$

ⓐ $\ln(5-2x)$

$$5-2x>0$$
$$5>2x$$
$$\frac{5}{2}>x \Rightarrow \left(-\infty, \frac{5}{2}\right)$$

ⓑ $\log(x^2-2x-15)$



$$x^2-2x-15>0$$
$$(x-5)(x+3)\boxed{>0}$$

$x-5=0$	$x+3=0$
$x=5$	$x=-3$

$(-4-5)(-4+3) = -9(-1) = +$

$-5 \cdot 3$

Domain: $(-\infty,-3) \cup (5,\infty)$

Properties of Logarithmic Functions

Because $f(x)=a^x$ and $f(x)=\log_a(x)$ are inverses

$$\log_a(a^x)=x$$

If $x>0$, $a^{\log_a(x)}=x$

If $a^x = a^y$, then $x = y$ b/c a^x is invertible (one-to-one)

Exponential Properties	Log properties
$a^b a^c = a^{b+c}$	$\log_a(b \cdot c) = \log_a(b) + \log_a(c)$
$\frac{a^b}{a^c} = a^{b-c}$	$\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$
$(a^b)^c = a^{b \cdot c}$	$\log_a(b^c) = c \log_a(b)$
$a^0 = 1$	$\log_a(1) = 0$
$a^1 = 1$	$\log_a(a) = 1$
$a^{-b} = \frac{1}{a^b}$	$\log_a\left(\underbrace{\tfrac{1}{b}}_{b^{-1}}\right) = \cancel{\log_a(1)}^{0} - \log_a(b) = -\log_a(b)$

Change of base formula: $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$

ex. $\log_a(b) = \frac{\log(b)}{\log(a)} = \frac{\ln(b)}{\ln(a)}$

Ex 4: Simplify the expression w/o using a calculator

(a) $\ln(e^2) = 2$ b/c ln and e are inverses

(b) $\log_2\left(\frac{1}{8}\right) = \log_2\left(\frac{1}{2^3}\right) = \log_2(2^{-3}) = -3$

(b) $\log_2\left(\frac{1}{8}\right) = \log_2\left(\frac{1}{2^3}\right) = \log_2\left(2^{-3}\right) = -3$

(c) $5^{\log_5(2)} = 2$ b/c $\log_5 x$ and 5^x are inverses

(d) $\log(1,000,000) = \log_{10}(1,000,000) = \log_{10}(10^6) = 6$

(e) $\log(0.001) = \log_{10}(10^{-3}) = -3$

(f) $e^{2\ln(7)} = \cancel{e}^{\cancel{\ln}(7)^2} = 7^2 = 49$

$2\ln(7) = \ln(7)^2$

(g) $\cancel{\ln(1)}^{\,0} + 4\cancel{\ln(e)}^{\,1} = 0 + 4 \cdot 1 = 4$

(h) $\log_3(\sqrt{27}) = \log_3(27^{1/2}) = \log_3((3^3)^{1/2}) = \log_3(3^{3/2}) = \frac{3}{2}$