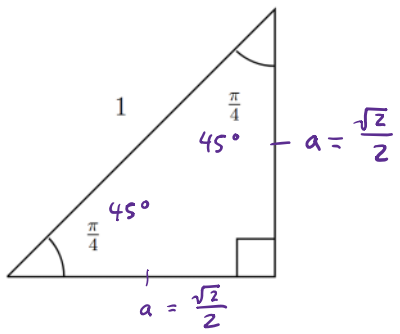
Lesson 25 - The Unit Circle

Two Special Right Triangles

The $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ or the $45° - 45° - 90°$ Triangle:



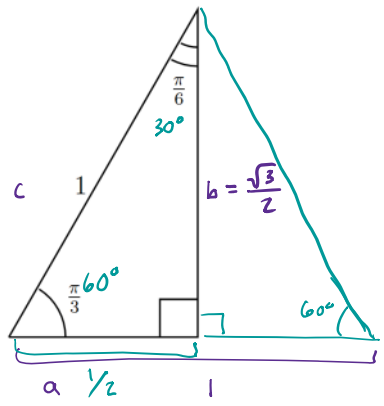
$$a^2 + a^2 = 1^2 = 1$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ or the $30° - 60° - 90°$ Triangle



Pythagorean thm: $a^2 + b^2 = c^2$

$$\left(\frac{1}{2}\right)^2 + b^2 = 1^2 = 1$$

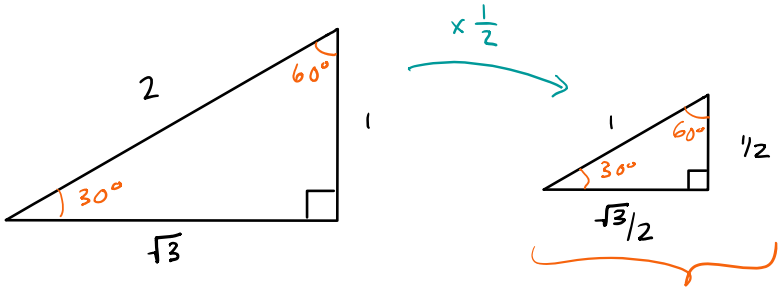
$$\frac{1}{4} + b^2 = 1 = \frac{4}{4}$$

$$b^2 = \frac{3}{4}$$

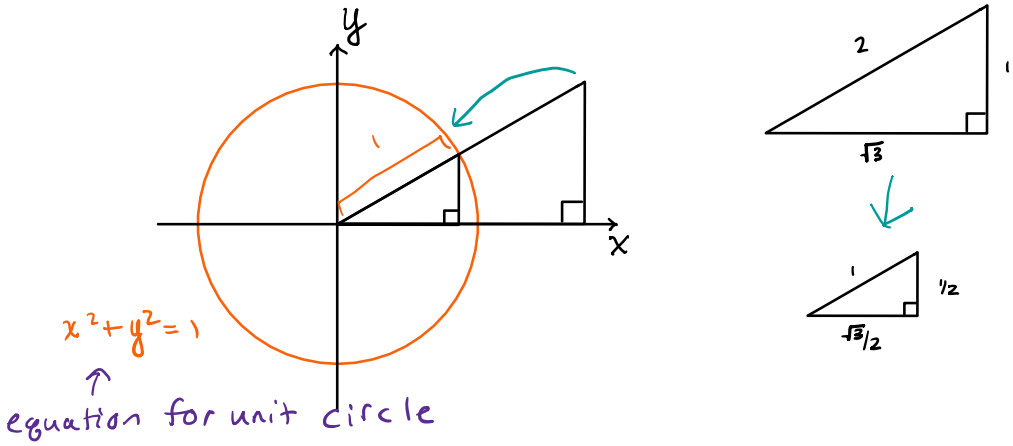
$$b = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

Unit Circle

- two triangles are similar if they have the same 3 angle measures
  (actually having two of the same angles is enough since $\alpha+\beta+\theta=180°$)
- similar triangles have the same ratios of side lengths

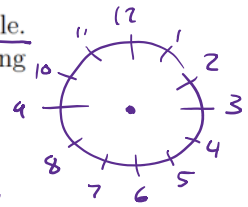


- Any two similar reference triangles have the same angles & trig ratios — only need one
  ↳ standardize (scale up or down) so r = hypotenuse length = 1



- it is called the unit circle b/c it has radius 1 (unit ↔ 1)

Example Let P be the intersection point of the terminal ray of the given angle and the unit circle. Determine the coordinates of P. Assume the angle is in standard position. Give exact answers, involving square roots if necessary.

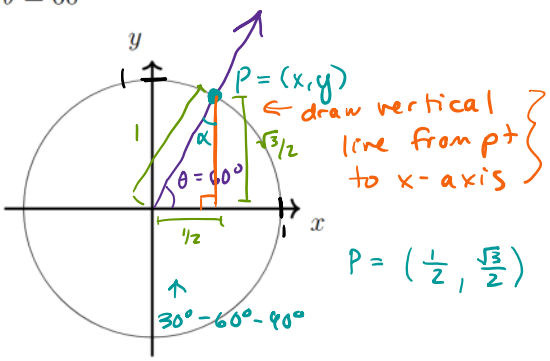


(a) $\theta = 60°$

$\frac{2}{3}$ of 90°

$60°+90°+\alpha=180°$

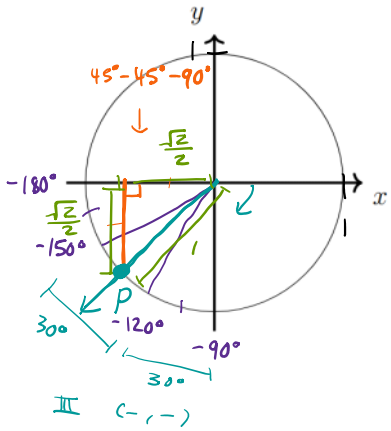
$\alpha = 30°$



$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

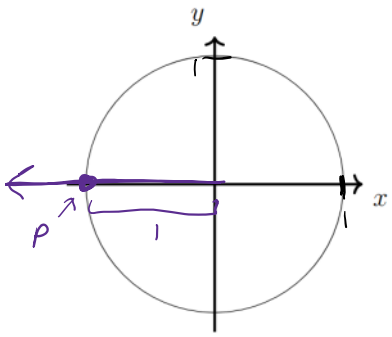
(b) $\theta = -135°$

$135° = 120° + 15°$



$P = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(c) $\theta = \pi$

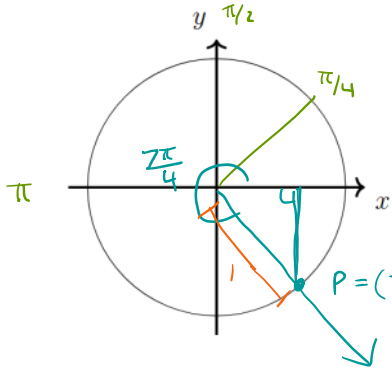


2π

π

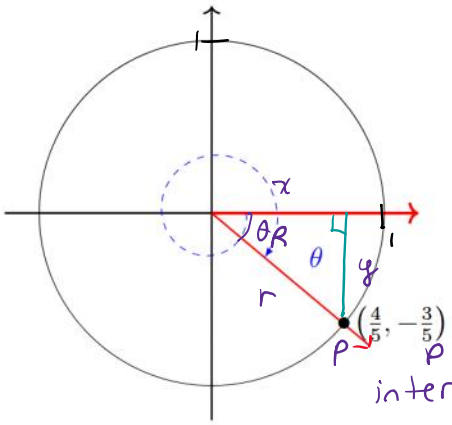
$P = (-1, 0)$

(d) $\theta = \frac{7\pi}{4}$ Find coords of P



QIV (+, −)

Example The angle θ is in standard position. Use the intersection point of the terminal ray and the unit circle to find the exact values of $\sin(\theta)$ and $\cos(\theta)$.



$\theta < -360°$

$\sin(\theta) = \frac{y}{r}$ $\xrightarrow[r=1]{\text{unit circle}}$ $\sin(\theta) = y$

$\cos(\theta) = \frac{x}{r}$ $\longrightarrow$ $\cos(\theta) = x$

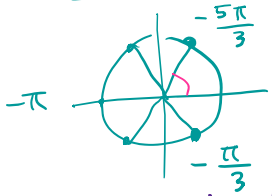
P is at intersection of terminal ray & unit circle

$r = 1$

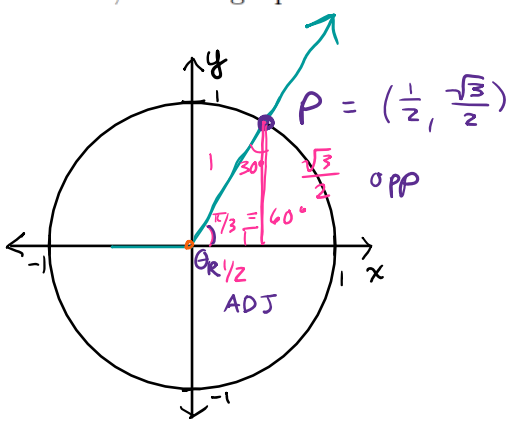
$\sin(\theta) = -\frac{3}{5}$

$\cos(\theta) = \frac{4}{5}$

Example Let P be the intersection point of the terminal ray of the given angle and the unit circle. Determine the coordinates of P. Assume the angle is in standard position. Then use this information to help you calculate the six trigonometric functions for the angle. Give exact answers, involving square roots if necessary.

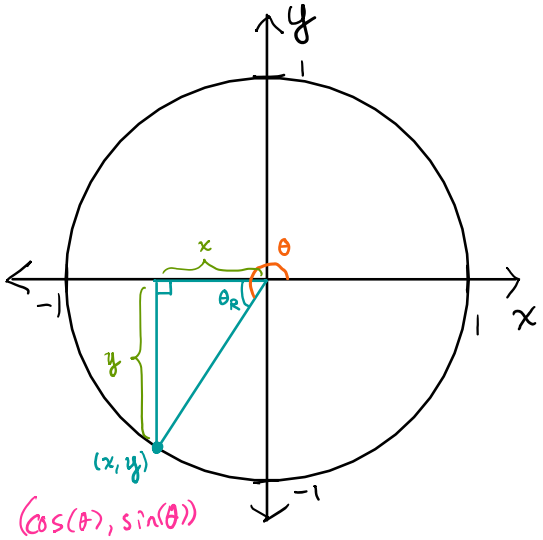


$\theta = \frac{-5\pi}{3}$



- $\sin(\theta) = \frac{y}{r} = \frac{\sqrt{3}}{2}$ $(= y)$ — reciprocal → $\csc(\theta) = \frac{1}{y} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
- $\cos(\theta) = \frac{1}{2}$ — $\sec(\theta) = \frac{1}{x} = 2$
- $\tan(\theta) = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$ — $\cot(\theta) = \frac{x}{y} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

slope of terminal ray



* $\sin(\theta) = y$ (vertical coordinate)

* $\cos(\theta) = x$ (horiz. coord.)

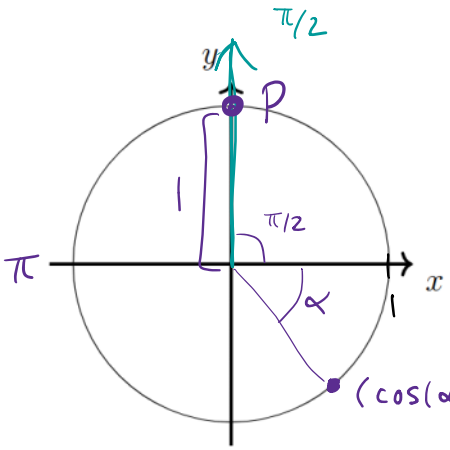
* $\tan(\theta) = \frac{y}{x}$ (slope of terminal ray)

$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y}$ * unless $y = 0$ (undefined)

$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$ * unless $x = 0$ (undefined)

$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y}$ * unless $y = 0$ (undef.)

Example Use the unit circle to find exact values for the following when $\theta = \frac{\pi}{2}$.



$x^2 + y^2 = 1$

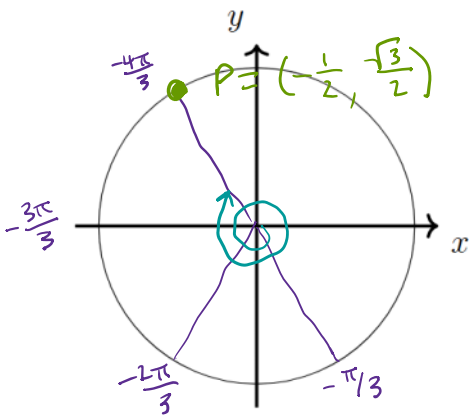
$0^2 + y^2 = 1$

$P = (0, 1)$

$x \quad y$

- $\sin(\theta) =$ $\sin\left(\frac{\pi}{2}\right) = 1$
- $\cos(\theta) =$ 0
- $\tan(\theta) =$ $\frac{y}{x} = \frac{1}{0} =$ UND
- $\csc(\theta) =$ $\frac{1}{y} = \frac{1}{1} = 1$
- $\sec(\theta) =$ $\frac{1}{x} = \frac{1}{0} =$ UND
- $\cot(\theta) =$ $\frac{x}{y} = 0$

Example Use the unit circle to find exact values for the following when $\theta = -\frac{10\pi}{3}$.



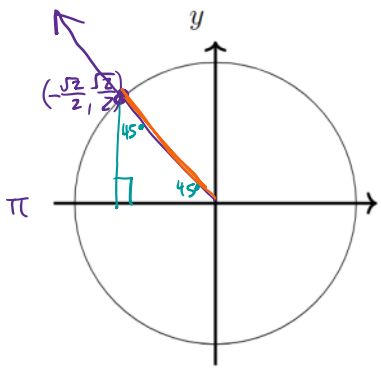
$2\pi = \frac{6\pi}{3}$

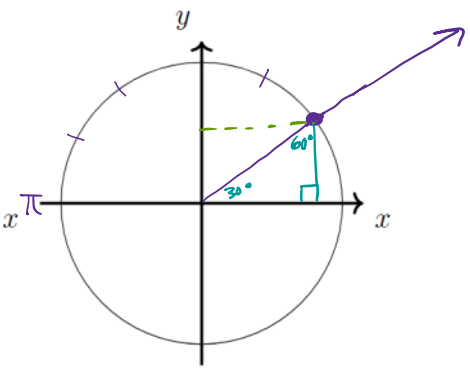
$\theta + 2\pi = -\frac{10\pi}{3} + \frac{6\pi}{3} = -\frac{4\pi}{3}$

- $\sin(\theta) = \frac{\sqrt{3}}{2}$
- $\csc(\theta) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
- $\cos(\theta) = -\frac{1}{2}$
- $\sec(\theta) = -2$
- $\tan(\theta) = -\sqrt{3}$
- $\cot(\theta) = -\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Example Find the exact value of $\tan\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$. $= \tan\left(\frac{3\pi}{4}\right) \cdot \sin\left(\frac{\pi}{6}\right)$

-1 $\quad \frac{1}{2}$





$= -\frac{1}{2}$