Lesson 33 - Systems of Equations

Systems of Equations $\begin{cases} f(x,y) = c \\ g(x,y) = d \end{cases}$

- a solution to a system of equations makes all equations in the system true at the same time

- two types of systems : $\begin{cases} \text{consistent} - \text{there exists } \underline{\text{at least one}} \text{ solution} \\ \text{inconsistent} - \underline{\text{no}} \text{ solutions exist} \end{cases}$

ex Are the following systems solved by the points given?

a) $\begin{cases} 2x + y = 13 & \rightarrow 2\cdot 3 + 7 = 6 + 7 = 13 \checkmark \\ y = 7 & \rightarrow \checkmark \end{cases}$

$(x,y) = (3,7) \Rightarrow$ yes, solution

b) $\begin{cases} 3x + 5y = 11 & 3\cdot 2 + 5\cdot 1 = 6 + 5 = 11 \checkmark \\ \underbrace{x - 2y}_{0} = 4 & 2 - 2\cdot 1 = 2 - 2 = 0 \neq 4 \times \end{cases}$

$(x,y) = (2,1) \Rightarrow$ no, not a solution

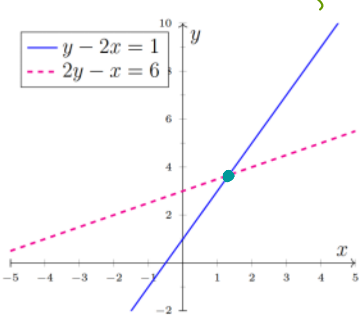
Systems of Linear Equations

- three types of linear systems :

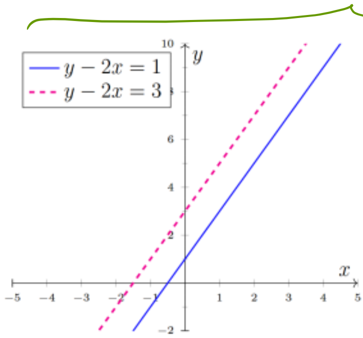
$$\begin{cases} \text{consistent} \begin{cases} \text{independent} - \text{exactly } \underline{\text{one}} \text{ solution} \\ \text{dependent} - \underline{\text{infinitely many}} \text{ solutions} \end{cases} \\ \text{inconsistent} - \underline{\text{no}} \text{ solutions exist} \end{cases}$$

different slopes

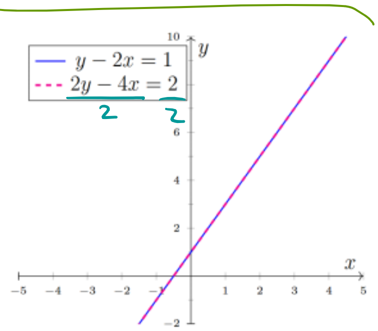
same slope



one solution
↓
consistent independent



no solutions
↓
inconsistent

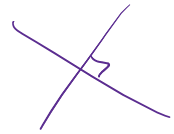


infinitely many solutions
↓
consistent dependent

system of 2 equations : solution $\longleftrightarrow$ graphs intersect

- In 2D, lines can be parallel, perpendicular, or neither

$$\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$$ m is slope



parallel if ... $m_1 = m_2$

90° angle ← perpendicular if ... $m_1 = -\frac{1}{m_2}$ aka $m_2 = -\frac{1}{m_1}$

or neither.

- solution by **substitution**: isolate one variable & substitute into other equation

ex

(a) $\begin{cases} y = x - 7 \\ 2x + y = 8 \end{cases}$ plug in

$2x + (x-7) = 8$

$2x + x - 7 = 8$

$3x = 15$

$x = 5$

$y = 5 - 7 = -2$

$(5, -2)$

$10 - 2 = 8$ ✓

one solution ⟷ consistent & independent

(b) $\begin{cases} x + 4y = 11 \\ 3x - 2y = 19 \end{cases}$ → $x = 11 - 4y$

$3(11-4y) - 2y = 19$

$33 - 12y - 2y = 19$

$33 - 19 = 12y + 2y$

$14 = 14y$

$y = 1$

works but is harder $\begin{cases} 3x = 19 + 2y \\ x = \frac{19}{3} + \frac{2}{3}y \end{cases}$

$\begin{array}{r} 33 \\ -19 \\ \hline 14 \end{array}$

$x = 11 - 4 \cdot 1$

$= 11 - 4$

$= 7$

$(7, 1)$

consistent & indep.

(c) $\begin{cases} 6x + y = 10 \\ y = -6x + 1 \end{cases}$ → $6x + (-6x+1) = 10$

$1 = 10$

this is NEVER true, so no solns. ⟷ inconsistent

(d) $\begin{cases} 2x + 4y = 8 \\ x = -2y + 4 \end{cases}$

$2(-2y+4) + 4y = 8$

$-4y + 8 + 4y = 8$

$8 = 8$

ALWAYS true → infinitely many solutions (consistent, dependent)

• solution by elimination: make both equations look like $ax+by=c$

add equations together $ax-ax=0$

$by-by=0$

ex

(a) $\begin{cases} 5x+3y=23 \\ x-3y=1 \end{cases}$

$$\begin{array}{r} 5x+3y=23 \\ +\quad x-3y=1 \\ \hline 6x+0y=24 \end{array}$$

$6x=24$

$x=4$

$4-3y=1$

$-3y=-3$

$y=1$

$(4,1)$

consistent, indep

(b) $\begin{cases} x+4y=11 \\ 3x-2y=19 \end{cases}$ $\xrightarrow{x\,-3}$

$$\begin{array}{r} -3x-12y=-33 \\ +\quad 3x-2y=19 \\ \hline -14y=-14 \end{array}$$

$y=1$

$x+4=11$

$x=7$

$(7,1)$

consistent, indep

(c) $\begin{cases} 6x+y=10 \\ y=-6x+1 \end{cases}$

$+6x \quad +6x$

$\to -1\cdot(6x+y)=(1)\cdot -1 \to -6x-y=-1$

$$\begin{array}{r} 6x+y=10 \\ +\quad -6x-y=-1 \\ \hline 0x+0y=9 \end{array}$$

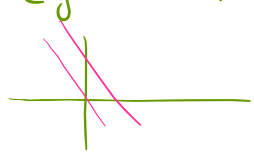
$0=9$

never true

no solutions

(inconsistent)

$\begin{cases} y=-6x+10 \\ y=-6x+1 \end{cases}$



(d) $\begin{cases} 2x+4y=8 \\ x=-2y+4 \end{cases}$ $\to x+2y=4 \xrightarrow{x(-2)}$

$$\begin{array}{r} 2x+4y=8 \\ +\quad -2x-4y=-8 \\ \hline 0x+0y=0 \end{array}$$

$0=0$

consistent, dependent $\Leftrightarrow$ infinitely many sol'ns $\leftarrow$ always true

at least one equation is not $ax+by=c$

Systems of Nonlinear Equations

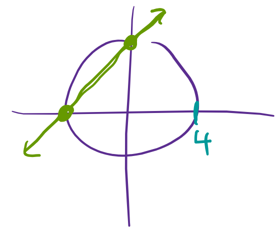
• there may be any number of solutions

• we still use substitution and elimination $\to$ only possible if there are like terms in both equations ex $x^2+y^2=r^2$

1. $\begin{cases} x^2+y^2=16 \\ x-y=-4 \end{cases}$ $\rightarrow$ circle of radius 4; $\rightarrow$ line $\rightarrow x = y-4$

substitution

isolate var. in 2nd eq.



$$(y-4)^2+y^2=16$$
$$y^2-8y+16+y^2=16$$
$$y^2-8y+y^2=0$$
$$2y^2-8y=0$$
$$y^2-4y=0$$
$$y(y-4)=0 \quad y=0,\; y=4 \rightarrow \text{need } x\text{-coord}$$

sub in to linear eqn

① $y=0 \rightarrow x-0=-4 \rightarrow x=-4$ $(-4,0)$

② $y=4 \rightarrow x-4=-4 \rightarrow x=0$ $(0,4)$

elimination

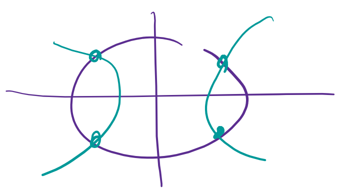
+ 2. $\begin{cases} x^2+y^2=49 \\ x^2-y^2=1 \end{cases}$ $\rightarrow$ hyperbola

$) \quad ($

$$2x^2=50$$
$$x^2=25$$
$$x=\pm\sqrt{25}=\pm5$$

$x=5 \xrightarrow{\text{1st eqn}} 25+y^2=49 \rightarrow y^2=24 \quad y=\pm\sqrt{24}$

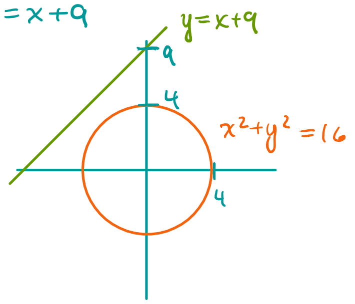
$x=-5 \quad 25+y^2=49 \longrightarrow y=\pm\sqrt{24}$



4 solns: $(5,\sqrt{24})$, $(5,-\sqrt{24})$, $(-5,\sqrt{24})$, $(-5,-\sqrt{24})$

3. $\begin{cases} x^2+y^2=16 \\ x-y=-9 \end{cases}$ $\rightarrow x=y-9$

$y=x+9$



$$(y-9)^2+y^2=16$$
$$y^2-18y+81+y^2=16$$
$$2y^2-18y+65=0 \quad a=2,\ b=-18,\ c=65$$
$$b^2-4ac = 18^2-4\cdot2\cdot65=-196$$

$\sqrt{-196}$ DNE $\rightarrow$ no solutions