Lesson 7: Function Composition

Last time, we learned how to combine f and g as

① $f+g$ ② $f-g$ ③ $f \cdot g$ ④ $\frac{f}{g}$

There is another way to combine these.

⑤ $f \circ g$ (Read "f composed with g")

$$(f \circ g)(x) \overset{\text{def}}{=} f(g(x))$$

Ex 1: Use the table to find the function values:

x	-2	0	1	3	7
f(x)	4	-5	8	0	5
g(x)	6	8	-2	7	-2

ⓐ $(f \circ g)(3) \overset{\text{def}}{=} f(g(3))$
$= f(7)$
$= 5$

ⓑ $(g \circ f)(3) \overset{\text{def}}{=} g(f(3))$
$= g(0)$
$= 8$

ⓒ $(g \circ g)(1) \overset{\text{def}}{=} g(g(1))$
$= g(-2)$
$= 6$

Note: In general, $(f \circ g)(x) \neq (g \circ f)(x)$ always

Domain of $f \circ g$: For x to be in the domain of $f \circ g$, we must have

ⓐ x is the domain of g

ⓑ $g(x)$ is the domain of f

$$(f \circ g)(x) = \underbrace{f(\underbrace{g(x)})}$$

Ex 2: Let $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x-1}$

ⓐ Find $f(g(1))$ and $g(f(1))$

$f(g(1)) = f(0) = \frac{1}{0} \Rightarrow$ undefined

$f(g(1)) = f(0) = \frac{1}{0} \Rightarrow$ undefined

$g(1) = \sqrt{1-1} = \sqrt{0} = 0$

$g(f(1)) = g(1) = \sqrt{1-1} = \sqrt{0} = 0$

$f(1) = \frac{1}{1} = 1$

(b) Find $(f \circ g)(x)$

$(f \circ g)(x) \underline{\underline{def}} f(g(x)) = f(\sqrt{x-1}) = \frac{1}{\sqrt{x-1}}$

If they want you to rationalize, $(f \circ g)(x) = \frac{1}{\sqrt{x-1}} \cdot \frac{\sqrt{x-1}}{\sqrt{x-1}} = \frac{\sqrt{x-1}}{x-1}$

(c) Domain of $(f \circ g)(x)$.

(i) Find domain of $g(x)$

(ii) Find domain of $(f \circ g)(x)$

Domain of g: $g(x) = \sqrt{x-1} \Rightarrow x-1 \geq 0$

$x \geq 1$

Domain of $f \circ g$: $\frac{1}{\sqrt{x-1}} \Rightarrow x-1 > 0$

$x > 1$

Where does $x \geq 1$ <u>and</u> $x > 1$ happens? Only happens @ $x > 1$ or $(1, \infty)$

<u>Ex 2:</u> Let $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x-1}$

(d) Find $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x}) = \sqrt{\frac{1}{x} - 1}$

<u>Rationalize:</u> $\sqrt{\frac{1}{x} - 1} = \sqrt{\frac{1}{x} - \frac{x}{x}} = \sqrt{\frac{1-x}{x}} = \frac{\sqrt{1-x}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{1-x} \cdot \sqrt{x}}{x} = \frac{\sqrt{x(1-x)}}{x}$

(e) Find the domain of $g \circ f$.

(i) Domain of f

(ii) Domain of $g \circ f$

Domain of f: $f(x) = \frac{1}{x} \Rightarrow x \neq 0$

<u>Domain of $g \circ f$:</u> $\sqrt{\frac{1}{x} - 1} \Rightarrow \frac{1}{x} - 1 \geq 0$

$\frac{1}{x} \geq 1$

This is true when $x > 0$ $\quad 1 \geq x > 0$

This is true when $x < 0$ $\quad 1 \leq x < 0$

$\underbrace{\qquad\qquad}_{0 < x \leq 1}$

When does $x \neq 0$ and $0 < x \leq 1$ happen at the same? $\underline{0 < x \leq 1}$ $(0, 1]$

Ex 3: Let $f(x) = \frac{1}{x}$

ⓐ Find $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = 1 \cdot \frac{x}{1} = x$

ⓑ Find the domain of $f \circ f$.

(i) Domain of f

(ii) Domain of $f \circ f$

Domain of f: $f(x) = \frac{1}{x} \Rightarrow x \neq 0$

Domain of $f \circ f$: $x \Rightarrow (-\infty, \infty)$

$\}$ Domain $= (-\infty, 0) \cup (0, \infty)$

Identity function: $i(x) = x$

For any function f, $(f \circ i)(x) = f(i(x)) = f(x)$

$(i \circ f)(x) = i(f(x)) = f(x)$

Ex 4: Find non-identity functions $f(x)$ and $g(x)$ so that $h(x) = f(g(x))$

ⓐ $h(x) = \frac{4}{x+1}$

Inside: $x+1 = g(x)$

Outside: $\frac{4}{x} = f(x)$

Check: $h(x) = f(g(x)) = f(x+1) = \frac{4}{x+1}$ ✓

ⓑ $h(x) = \sqrt{x^3 - 2} + 5$

$\boxed{\begin{array}{l} \text{Inside: } x^3 - 2 = g(x) \\ \text{Outside: } \sqrt{x} + 5 = f(x) \end{array}}$

BUT Valid

Inside: x^3

Outside: $\sqrt{x-2} + 5$

BUT Inside: $\sqrt{x^3 - 2}$

Outside: $x + 5$

Ex 5: A rock is dropped into a pond creating circular ripples. The radius of the outer circle is increasing at a rate of 3 ft/sec.

ⓐ Write down a function $r(t)$ that expresses the radius as a function of time.

$$r(t) = \left(3 \frac{ft}{\cancel{s}}\right) + \cancel{s} = 3t \quad ft$$

ⓑ If a circle has radius r, what is its area A?

$A = \pi r^2$

(b) If a circle has radius r, what is its area A?

$$A = \pi r^2$$

$$A(r) = \pi r^2$$

(c) Write down a function $A(t)$ that expresses the area of the outer circle as a function of time.

$$A(t) = (A \circ r)(t) = A(r(t)) = A(3t) = \pi (3t)^2 = \pi \cdot 9t^2 = 9\pi t^2$$