No class 2/13

Written HW 3 Due Friday (In-class)

Lesson 9: Average Rate of Change

Average Rate of Change

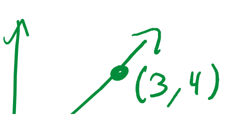
- When two quantities change together, we can compute the rate of change:
  - q_1 = quantity 1
  - q_2 = quantity 2
  - Δq_1 = change in quantity 1
  - Δq_2 = change in quantity 2
  - $\dfrac{\Delta q_1}{\Delta q_2}$ is the rate of change of quantity 1 with respect to quantity 2
- For a function $y=f(x)$, we calculate the quantity

$$\frac{\Delta y}{\Delta x} = \underbrace{\frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}}_{\text{Slope}} = \frac{\Delta f(x)}{\Delta x}$$

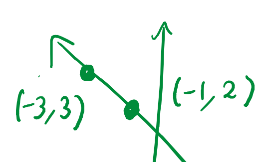
- Rate of change may be different for different choices of x_1 and x_2
- Since Rate of change could fluctuate on $[x_1, x_2]$ we call $\dfrac{y_2-y_1}{x_2-x_1}$ the <u>average rate of change</u>
- $\dfrac{y_2-y_1}{x_2-x_1}$ is also the slope of the line connecting (x_1, y_1) and (x_2, y_2)

Ex 1: Find the average rate of change from the given graph

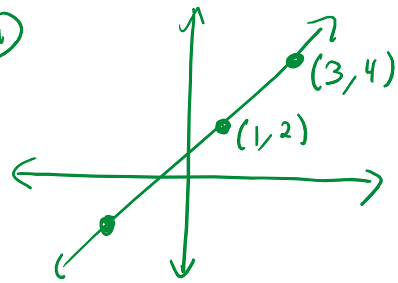
(a)



(b)



(a)



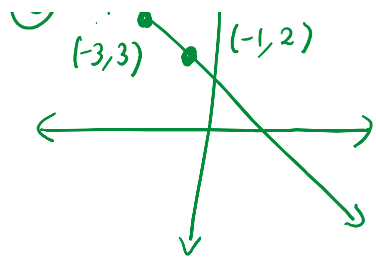
Average rate of change can be found with any 2 points. Choose 2 to find it

(3,4) and (1,2)

$x_1 = 3$ $\quad$ $x_2 = 1$

$y_1 = 4$ $\quad$ $y_2 = 2$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{1-3} = \frac{-2}{-2} = 1$$



(−3,3) $\quad$ (−1,2)

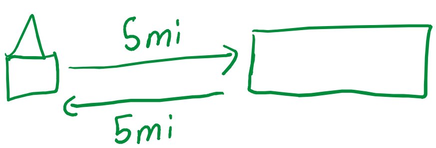
$x_1 = -3$ $\quad$ $x_2 = -1$

$y_1 = 3$ $\quad$ $y_2 = 2$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{-1+(+3)} = \frac{-1}{2} = -\frac{1}{2}$$

Ex 2: Joni leaves the house @ 9am and drives to the store, which is 5 miles away. She returns from her trip @ 11:30am (not having made any other stops). What was the average speed of Joni's car from 9 am to 11:30 am?

$$\text{Speed} = \frac{\Delta \text{ distance}}{\Delta \text{ time}} = \frac{5mi + 5mi}{2.6 \text{ hrs}} = \frac{10 \text{ mi}}{2.5 \text{ hrs}} = 4 \frac{mi}{hr}$$



Ex 3: A boat launches @ 5pm. The total distance it has traveled t hours after 5pm is given by $d(t) = 5t^3$ where $d(t)$ is measured in miles. What is the average speed of the boat between

(a) 5 pm and 6pm?

$\Downarrow$ $\quad$ $\Downarrow$

$t=0$ $\quad$ $t=1$

$$\text{Speed} = \frac{d(1) - d(0)}{1-0} = \frac{5(1)^3 - 5(0)}{1-0} = \frac{5}{1} = 5 \frac{mi}{hr}$$

(b) 6pm and 7pm?

$\Downarrow$ $t=1$ $\quad$ $\Downarrow$ $t=2$

$$\text{Speed} = \frac{d(2)-d(1)}{2-1} = \frac{5(2)^3-5(1)^3}{2-1} = \frac{5\cdot 8-5\cdot 1}{1}$$

$$= \frac{40-5}{1} = 35\ \frac{mi}{hr}$$

Ex 4: Given

x	-3	1	2	4	5	10
$f(x)$	4	-2	20	-1	3	2

Use the table to calculate the average rate of change of $f(x)$ on the interval

(a) $[2,5]$

$$\text{Average} = \frac{f(5)-f(2)}{5-2} = \frac{3-20}{5-2} = -\frac{17}{3}$$

(b) $[1,10]$

$$\text{Average} = \frac{f(10)-f(1)}{10-1} = \frac{2-(-2)}{10-1} = \frac{2+2}{10-1} = \frac{4}{9}$$

Ex 5: Let $f(x) = 2x^3$.

(a) Compute the average rate of change of $f(x)$ over the interval $[a,4]$ (Assume $a \neq 4$).

$$\text{Average} = \frac{f(4)-f(a)}{4-a} = \frac{2(4)^3-2a^3}{4-a} = \frac{2[4^3-a^3]}{4-a}$$

Remember $a^3-b^3 = (a-b)(a^2+ab+b^2)$

$$= \frac{2\cancel{(4-a)}(4^2+4\cdot a+a^2)}{\cancel{4-a}} = 2(16+4a+a^2)$$

(b) Find all a such that the average rate of change of $f(x)$ on the interval $[a,4]$ is 26.

$$\text{Average} = 26$$

By part a,

Average $= 26$

By part a,

$$\frac{2(16+4a+a^2)}{2} = \frac{26}{2}$$

$$a^2+4a+16 = 13$$
$$-13 \quad -13$$

$$a^2+4a+3=0$$
$$a^2+3a+a+3=0$$
$$a(a+3)+1(a+3)=0$$
$$(a+1)(a+3)=0$$
$$a=-1,-3$$

3
$1+3=4$

Difference Quotient

- Given a function f, the difference quotient of f(x) is

$$\star \quad \frac{f(x)-f(a)}{x-a} \quad \text{or more commonly} \quad \frac{f(x+h)-f(x)}{h}$$

Assume $h \neq 0$.

Ex 6: Find the average rate of change of the function on the given interval. Simplify your answer.

ⓐ $g(x) = x^2+3x+5$ on the interval $[x, x+h]$ where $h \neq 0$.

$$\frac{g(x+h)-g(x)}{(x+h)-x} = \frac{(x+h)^2+3(x+h)+5-(x^2+3x+5)}{h}$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$= \frac{\cancel{x^2}+2xh+h^2+\cancel{3x}+3h+\cancel{5}-\cancel{x^2}-\cancel{3x}-\cancel{5}}{h}$$

$$= \frac{2xh+h^2+3h}{h}$$

$$= \frac{\cancel{h}(2x+h+3)}{\cancel{h}} = 2x+h+3$$

(b) $m(x) = 9$ on the interval $[x, x+h]$ where $h \neq 0$

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$$\frac{m(x+h) - m(x)}{(x+h) - x} = \frac{9-9}{h} = \frac{0}{h} = 0$$

(c) $g(x) = \frac{1}{2x+1}$ on the interval $[t, t+h]$ where $h \neq 0$

$$\frac{g(t+h) - g(t)}{(t+h) - t} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{\frac{1}{2t+2h+1} - \frac{1}{2t+1}}{h}$$

$$= \frac{\frac{1}{2t+2h+1} \cdot \frac{(2t+1)}{(2t+1)} - \frac{1}{2t+1} \cdot \frac{(2t+2h+1)}{(2t+2h+1)}}{h}$$

$$= \frac{\frac{\cancel{2t} + \cancel{1} - (\cancel{2t} + 2h \cancel{+1})}{(2t+2h+1)(2t+1)}}{h}$$

$$= \frac{\frac{-2h}{(2t+2h+1)(2t+1)}}{h/1}$$

$$= \frac{-2\cancel{h}}{(2t+2h+1)(2t+1)} \cdot \frac{1}{\cancel{h}} = \frac{-2}{(2t+2h+1)(2t+1)}$$

(d) $k(x) = \sqrt{2+x}$ on the interval $[x, x+h]$ where $h \neq 0$

Trick: is using the conjugate.

$$\frac{k(x+h) - k(x)}{(x+h) - x} = \frac{\sqrt{2+x+h} - \sqrt{2+x}}{h}$$

$$= \frac{(\sqrt{2+x+h} - \sqrt{2+x})}{h} \cdot \frac{(\sqrt{2+x+h} + \sqrt{2+x})}{(\sqrt{2+x+h} + \sqrt{2+x})}$$

$$(a-b)(a+b) = a^2 - b^2$$

$(a-b)(a+b) = a - b$

$$= \frac{\left(\sqrt{2+x+h}\right)^2 - \left(\sqrt{2+x}\right)^2}{h\left(\sqrt{2+x+h} + \sqrt{2+x}\right)}$$

$$= \frac{\cancel{2} + \cancel{x} + h - (\cancel{2} + \cancel{x})}{h\left(\sqrt{2+x+h} + \sqrt{2+x}\right)}$$

$$= \frac{\cancel{h} \cdot 1}{\cancel{h}\left(\sqrt{2+x+h} + \sqrt{2+x}\right)}$$

$$= \frac{1}{\sqrt{2+x+h} - \sqrt{2+x}}$$