Lesson 10: The Chain Rule

Recall composition of functions

Let $y = f(g(x))$ $\Rightarrow$ g - inner function
f - outer function

Ex 1: Determine f and g for $y = (3x+1)^2$

$f(x) = (x)^2$ $g(x) = 3x+1$

Check $y = f(g(x)) = f(3x+1) = (3x+1)^2$ ✓

Ex 2: Determine f and g for $y = \sin^2 x = (\sin x)^2$

$f(x) = x^2$ $g(x) = \sin(x)$

Check $y = f(g(x)) = f(\sin x) = (\sin x)^2$ ✓

Ex 3: Determine f and g for $y = \tan(3x)$

$f(x) = \tan(x)$ $g(x) = 3x$

Ex 4: Determine f and g for $y = \sqrt[3]{2x+1}$

$f(x) = \sqrt[3]{x}$ $g(x) = 2x+1$

Chain Rule

Let $y = f(g(x))$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Ex 1: Find y' of $y=(3x+1)^2$.

Method 1: Expand and then use power rule

Recall $(a+b)^2 = a^2 + 2ab + b^2$

So $y = (3x)^2 + 2(3x)(1) + 1^2$

$= 9x^2 + 6x + 1$

$y' = 9(2)x^1 + 6$

$= 18x + 6$

Method 2: Chain Rule

$f(x) = x^2$ $\quad\quad$ $g(x) = 3x+1$

$f'(x) = 2x$ $\quad\quad$ $g'(x) = 3$

By chain rule,

$y' = f'(g(x)) \cdot g'(x)$

$= f'(3x+1) \cdot 3$

$= 2(3x+1) \cdot 3$

$= 6(3x+1)$

$= 18x + 6$

Ex 2: Find y' of $y = (3x^2+2x+1)^7$

Let $f(x) = x^7$ $\quad\quad$ $g(x) = 3x^2+2x+1$

$f'(x) = 7x^6$ $\quad\quad$ $g'(x) = 6x+2$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(3x^2 + 2x + 1) \cdot (6x + 2)$$
$$= 7(3x^2 + 2x + 1)^6 (6x + 2)$$

Ex 3: Find y' of $y = 2\cos^3 x$.
$$= 2(\cos x)^3$$

Let $f(x) = 2x^3$ $\quad g(x) = \cos x$
$f'(x) = 6x^2$ $\quad g'(x) = -\sin x$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(\cos(x)) \cdot (-\sin x)$$
$$= 6(\cos x)^2 (-\sin x)$$
$$= -6\cos^2 x \sin x$$

Ex 4: Find y' of $y = 5\tan(e^{3x})$

Let $f(x) = 5\tan(x)$ $\quad g(x) = e^{3x}$ ← To find this I need another chain rule.

$h(x) = e^x$ $\quad j(x) = 3x$
$h'(x) = e^x$ $\quad j'(x) = 3$

$f'(x) = 5\sec^2(x)$ $\quad g'(x) = h'(j(x)) \cdot j'(x)$
$$= h'(3x) \cdot 3$$
$$= e^{3x} \cdot 3$$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(e^{3x}) \cdot 3e^{3x}$$

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(e^{3x}) \cdot 3e^{3x}$$
$$= 5\sec^2(e^{3x}) \cdot 3e^{3x}$$
$$= 15e^{3x}\sec^2(e^{3x})$$

Ex 5: Find y' of $y = \left(\frac{2x}{3x^2+x}\right)^3$

Rewrite: $y = \left(\frac{2\cancel{x}}{\cancel{x}(3x+1)}\right)^3$
$$= \left(\frac{2}{3x+1}\right)^3$$
$$= \frac{2^3}{(3x+1)^3}$$
$$= \frac{8}{(3x+1)^3}$$
$$= 8(3x+1)^{-3}$$

Let $f(x) = 8x^{-3}$ $\quad g(x) = 3x+1$
$f'(x) = 8(-3)x^{-4}$ $\quad g'(x) = 3$
$= -24x^{-4}$

By chain rule,
$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(3x+1) \cdot 3$$
$$= -24(3x+1)^{-4} \cdot 3$$
$$= \frac{-72}{(3x+1)^4}$$

Ex 6: Given $y = \frac{15}{\sqrt[3]{x^2+1}}$. Find $y'(1)$.

Rewrite so no quotient rule.
$$y = \frac{15}{(x^2+1)^{1/3}} = 15(x^2+1)^{-1/3}$$

Let $f(x) = 15x^{-1/3}$ $\quad g(x) = x^2+1$
$f'(x) = 15\left(\frac{-1}{3}\right)x^{-1/3-1}$ $\quad g'(x) = 2x$
$= -5x^{-4/3}$

By chain rule,

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(x^2+1) \cdot 2x$$
$$= -5(x^2+1)^{-4/3}(2x)$$
$$= \frac{-5 \cdot 2x}{(x^2+1)^{4/3}} = \frac{-10x}{(x^2+1)^{4/3}}$$

I want $y'(1)$

$$y'(1) = \frac{-10(1)}{(1^2+1)^{4/3}} = \frac{-10}{2^{4/3}} = \frac{-10}{2^{3/3}2^{1/3}} = \frac{-10}{2 \cdot \sqrt[3]{2}} = \frac{-5}{\sqrt[3]{2}}$$