Lesson 13: Implicit Differentiation

Explicit Form: $y = f(x)$

Implicit Form: When a function is NOT written in explicit form

ex. ① $y - 2x = 1$ ② $x^2 + y^2 = 2$ ③ $y^2 + y - 1 = x$

To differentiate functions of this kind, we use a technique called implicit differentiation.

Use this technique when solving for y is MESSY.

Ex 1: Use implicit differentiation to find slope of tangent line of

$$x^2 - y^2 = 4x + 8y \quad \text{at } (0,0)$$

i.e. $\frac{dy}{dx}(0,0)$

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4x + 8y)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(8y)$$

$$2x \cancelto{1}{\frac{dx}{dx}} - 2y\frac{dy}{dx} = 4\cancelto{1}{\frac{dx}{dx}} + 8\frac{dy}{dx}$$

$$2x - 2y\frac{dy}{dx} = 4 + 8\frac{dy}{dx}$$

Solve for dy/dx.

$$2x - 4 = 8\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2x - 4 = (8 + 2y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x-4}{8+2y} \qquad \Rightarrow \frac{dy}{dx}(0,0) = \frac{2(0)-4}{8+2(0)} = \frac{-4}{8} = -\frac{1}{2}$$

Ex 2: Use implicit differentiation to find dy/dx of

$$yx^2 + e^y = x$$

$$\frac{d}{dx}(yx^2 + e^y) = \frac{d}{dx}(x)$$

$$\underbrace{\frac{d}{dx}(yx^2)} + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

Is this a function of x or y? Both

Is it a product or quotient? Product $\Rightarrow$ Product Rule

$$\frac{d}{dx}(y)\cdot x^2 + y\cdot\frac{d}{dx}(x^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$1\cdot\frac{dy}{dx}\cdot x^2 + y\cdot 2x\cancel{\frac{dx}{dx}}^{1} + e^y\frac{dy}{dx} = 1\cdot\cancel{\frac{dx}{dx}}^{1}$$

$$x^2\frac{dy}{dx} + 2xy + e^y\frac{dy}{dx} = 1$$

Solve dy/dx

$$x^2\frac{dy}{dx} + e^y\frac{dy}{dx} = 1 - 2xy$$

$$(x^2 + e^y)\frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{1-2xy}{x^2+e^y}$$

Extra Credit 2

Determine

ⓐ $\frac{d}{dx}(xy) = \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)$ ⓑ $\frac{d}{dx}\left(\frac{x}{y}\right)$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x)\, y + x \frac{d}{dx}(y)$$

$$= 1 \cdot \cancel{\frac{dx}{dx}}^{\,1} \cdot y + x \cdot 1 \cdot \frac{dy}{dx}$$

$$\underbrace{= y + x\frac{dy}{dx}}_{\text{Basically product rule}}$$

$$\frac{d}{dx}\left(\frac{x}{y}\right)$$

Hint: Quotient Rule

Ex 3: Use implicit differentiation to find dy/dx of

$$4\sin(x)\cos(y) = 3$$

$$\frac{d}{dx}(4\sin(x)\cos(y)) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(4\sin(x))\cos(y) + 4\sin(x)\cdot\frac{d}{dx}(\cos(y)) = \frac{d}{dx}(3)$$

$$4\cos(x)\cancel{\frac{dx}{dx}}^{\,1}\cos(y) + 4\sin(x)[-\sin(y)]\frac{dy}{dx} = 0$$

$$4\cos(x)\cos(y) = 4\sin(x)\sin(y)\frac{dy}{dx}$$

$$\frac{4\cos(x)\cos(y)}{4\sin(x)\sin(y)} = \frac{dy}{dx}$$

$$\cot(x)\cot(y) = \frac{dy}{dx}$$

Ex 4: Use implicit differentiation to find dy/dx of

$$6\tan(2x+3y) = 11x$$

$$\underbrace{\frac{d}{dx}(6\tan(2x+3y))}_{\text{Chain Rule}} = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\left[2\cancelto{1}{\frac{dx}{dx}} + 3\frac{dy}{dx}\right] = 11\cancelto{1}{\frac{dx}{dx}}$$

$$6\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = 11$$

$$2 + 3\frac{dy}{dx} = \frac{11}{6\sec^2(2x+3y)}$$

$$2 + 3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y)$$

$$3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y) - 2$$

$$\frac{dy}{dx} = \frac{11}{18}\cos^2(2x+3y) - \frac{2}{3}$$

Formal Proof of why $\frac{d}{dx}[\ln x] = \frac{1}{x}$

Let $y = \ln x$. Note $y = \ln x \iff e^y = x$

Differentiate

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx} = 1 \cdot \frac{dx}{dx}$$

$$e^y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Hence $\frac{d}{dx}[\ln x] = \frac{1}{x}$