Lesson 1: Review

Exponential Rules

1. $x^a x^b = x^{a+b}$
2. $\dfrac{x^a}{x^b} = x^{a-b}$
3. $(x^a)^b = x^{ab}$
4. $x^1 = x$
5. $x^0 = 1$
6. $x^{-1} = 1/x$

Logarithmic Rules

1. $\ln(1) = 0$
2. $\ln(e^x) = x$
3. $e^{\ln(x)} = x$
4. $\ln(xy) = \ln(x) + \ln(y)$
5. $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$
6. $\ln(x^m) = m\ln(x)$

Trig Identities

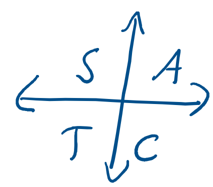
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\csc\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	$0 = \sqrt{0}/2$	$1/2 = \sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2 = 1$
cos	$1 = \sqrt{4}/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2 = \sqrt{1}/2$	$\sqrt{0}/2 = 0$
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined



Lesson 2:

① Finding Limits Numerically

Definition: If $f(x)$ approaches ($\to$) L as $x \to c$ we say that the limit of $f(x)$ as $x \to c$ is L.

i.e. $\lim_{x \to c} f(x) = L$

Note: That f does not need to be defined @ $x = c$ for the limit to exist.

Definition: If $f(x)$ increases or decreases without bound as $x \to c$, Definition: If $f(x)$ increases or decreases without bound as $x \to c$, then $\lim_{x\to c}$ is an infinite limit.

- If $f(x)$ increases without bound, $\lim_{x\to c} f(x) = +\infty$
- If $f(x)$ decreases without bound $\lim_{x\to c} f(x) = -\infty$

So when we have $\lim_{x\to c} f(x)$, we evaluate $f(x)$ at values of x that are getting closer and closer to c and see what happens with the values of the values of the function.

Ex 1: Evaluate $\lim_{x\to 4} (2x-3)$ numerically.

Recall $\lim_{x\to c} f(x)$. What is $f(x)$? $\underline{2x-3} = f(x)$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
f(x)	4.8	4.98	4.998	—	5.002	5.02	5.2

$\longrightarrow 5 \longleftarrow$

Hence $\lim_{x\to 4} (2x-3) = 5$

Ex 2: Evaluate $\underbrace{\lim_{x\to 3} \frac{x^3-3x^2}{x-3}}$ numerically.

$f(x) = \frac{x^3-3x^2}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
f(x)	8.41	8.9401	8.9940	—	9.006	9.0601	9.61

$\longrightarrow 9 \longleftarrow$

Hence $\lim_{x\to 3} \frac{x^3-3x^2}{x-3} = 9$

Ex 3: Given $f(x) = \begin{cases} x^2+1 & \text{if } x \neq -4 \\ 2 & \text{if } x = -4 \end{cases}$. Evaluate $\lim_{x\to -4} f(x)$ numerically

What is $f(x)$ when $x \neq -4$? $f(x) = x^2+1$

x	-4.1	-4.01	-4.001	-4	-3.999	-3.99	-3.9
$f(x)$	17.81	17.0801	17.0080	—	16.992	16.9201	16.21

$\longrightarrow 17 \longleftarrow$

$\lim_{x\to -4} f(x) = 17$ Note $f(-4) = 2$. So $\lim_{x\to -4} f(x) \neq 2$

Moral: $\lim_{x\to c} f(x)$ doesn't necessarily equal $f(c)$

Ex 4: Evaluate $\underbrace{\lim_{x\to 0} \frac{1}{x}}_{f(x)=1/x}$ numerically

No match $\Rightarrow \lim_{x\to 0} \frac{1}{x} = \text{DNE}$

② One-Sided Limit

Definition: A one-sided limit is the value that the function on $f(x) \to L$ as $x \to c$ from the left or right.

Left-Sided Limit: $\lim_{x\to c^-} f(x) = L$

Left-Sided Limit: $\lim_{x \to c^-} f(x) = L$

Right-Sided Limit: $\lim_{x \to c^+} f(x) = L$

Ex 5: What is $\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0^+} \frac{1}{x}$ numerically?

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-10	-100	-1000	$-$	1000	100	10

$\longrightarrow$ 9544563 $\longleftarrow$

$-\infty$ ∞

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$$

Lesson 3: Finding Limits Analytically

There are 3 different cases to consider.

① $f(c)$ returns a # (it could be 0)

i.e. $f(x)$ is continuous @ $x=c$

i.e. $\lim\limits_{x\to c} f(x) = f(c)$

Ex 1: $\lim\limits_{x\to 4}(2x-3) = 2(4)-3 = 8-3 = 5$

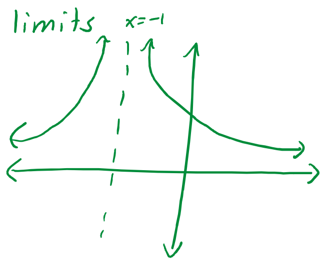
② $f(c)$ returns $\dfrac{\text{nonzero }\#}{0}$

i.e. Vertical Asymptote @ $x=c$

i.e. $\lim\limits_{x\to c} f(x) = \pm\infty$ or DNE

Ex 2: $\lim\limits_{x\to -1} \dfrac{1}{(x+1)^2}$

$f(-1) = \dfrac{1}{(-1+1)^2} = \dfrac{1}{0} \Rightarrow$ We need to check the left and right limits



$$\left.\begin{aligned} \lim_{x\to -1^-} \frac{1}{(x+1)^2} &= \infty \\ \lim_{x\to -1^+} \frac{1}{(x+1)^2} &= \infty \end{aligned}\right\} \Rightarrow \lim_{x\to -1} \frac{1}{(x+1)^2} = \infty$$

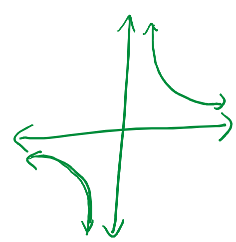
Ex 3: $\lim\limits_{x\to -1} \dfrac{-1}{(x+1)^2} = -\left[\lim\limits_{x\to -1} \dfrac{1}{(x+1)^2}\right] = -\infty$ by Ex 2

Ex 4: $\lim\limits_{x\to 0} \dfrac{1}{x}$

$f(0) = \dfrac{1}{0} \Rightarrow$ We need to check the left and right limits

$$\left.\begin{aligned} \lim_{x\to 0^-} \frac{1}{x} &= -\infty \\ \lim_{x\to 0^+} \frac{1}{x} &= \infty \end{aligned}\right\}$$ BUT they don't match

$\lim\limits_{x\to 0} \dfrac{1}{x} = \text{DNE}$



③ $f(c)$ returns $\dfrac{0}{0}$

i.e. $f(x)$ has a hole (if a factor cancels out) or VA @ $x=c$ (no factor cancels)

(3) $f(c)$ returns $\frac{0}{0}$

i.e. $f(x)$ has a hole (if a factor cancels out) or VA @ $x=c$ (no factor cancels)

Ex 5: $\lim\limits_{x\to 3} \dfrac{x^3-3x^2}{x-3}$

$f(3) = \dfrac{3^3-3\cdot 3^2}{3-3} = \dfrac{27-27}{3-3} = \dfrac{0}{0} \Rightarrow$ Factor!

$\lim\limits_{x\to 3} \dfrac{x^3-3x^2}{x-3} = \lim\limits_{x\to 3} \dfrac{x^2\cancel{(x-3)}}{\cancel{x-3}} = \lim\limits_{x\to 3} x^2 = 3^2 = 9$

Ex 6: $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

$f(4) = \dfrac{\sqrt{4}-2}{4-4} = \dfrac{2-2}{4-4} = \dfrac{0}{0} \Rightarrow$ Factor!

$= \lim\limits_{x\to 4} \dfrac{\cancel{\sqrt{x}-2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}}$

$= \lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2}$

$= \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{2+2} = \dfrac{1}{4}$

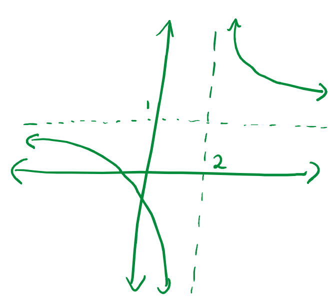
Difference of Squares
$a^2-b^2 = (a+b)(a-b)$

$x-4 = x-2^2$
$= (\sqrt{x})^2 - 2^2$
$= (\sqrt{x}+2)(\sqrt{x}-2)$

Ex 7: $\lim\limits_{x\to 2} \dfrac{x^2-2x}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{x\cancel{(x-2)}}{(x-2)^{\cancel{2}}} = \lim\limits_{x\to 2} \dfrac{x}{x-2} = \dfrac{2}{2-2} = \dfrac{2}{0}$

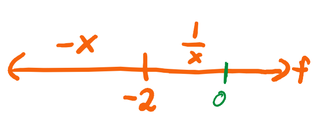
But now it looks like Case 2. So we need to check the left and right limits.

$\dfrac{x}{x-2} = \dfrac{x-2+2}{x-2} = \dfrac{x-2}{x-2} + \dfrac{2}{x-2} = 1 + \dfrac{2}{x-2}$

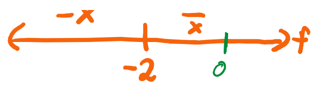


By the graph,
$\lim\limits_{x\to 2} \dfrac{x^2-2x}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{x}{x-2} = \text{DNE}$

Ex 8: $f(x) = \begin{cases} 1/x & \text{if } x \geq -2 \\ -x & \text{if } x < -2 \end{cases}$



$\{ -x$ if $x<-2$



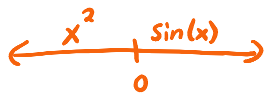
ⓐ $\lim_{x \to -2} f(x)$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} -x = -(-2) = 2$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} \frac{1}{x} = \frac{1}{-2}$ $\neq$ DNE

ⓑ $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x} = DNE$

Ex 9: $f(x) = \begin{cases} \sin(x) & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$



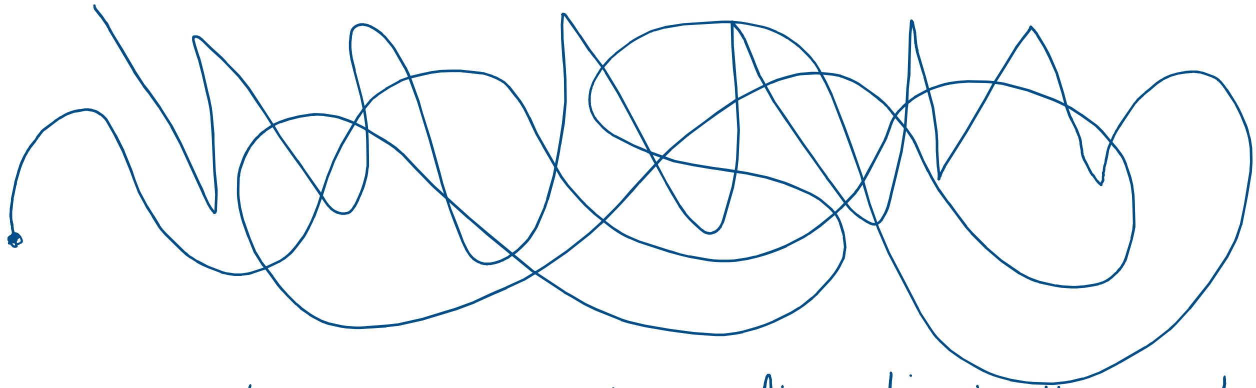
Find $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(x) = \sin(0) = 0$

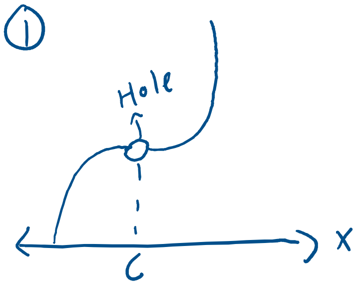
✓ $\lim_{x \to 0} f(x) = 0$

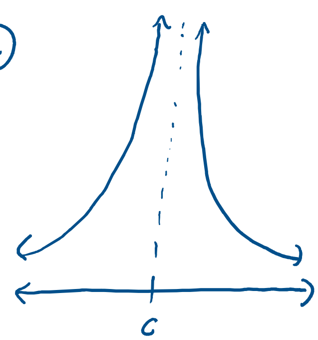
Lesson 4: Continuity



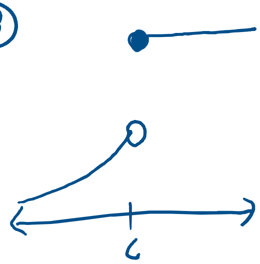
A function is continuous if there is no disruption in the graph.

The following 4 graphs show $f(x)$ is discontinuous @ $x=c$.

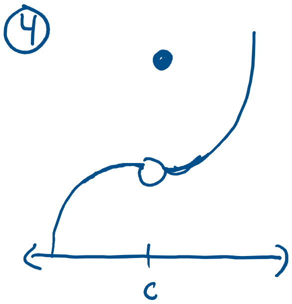
① 

② 

These 2 graphs have $f(c)$ undefined.

③ 

$f(c)$ is defined
BUT $\lim_{x \to c} f(x) = \text{DNE}$

④ 

$f(c)$ is defined
$\lim_{x \to c} f(c)$ exists
BUT $\lim_{x \to c} f(x) \neq f(c)$

We can see a function $f(x)$ is continuous @ $x=c$ if the following is true:

① $f(c)$ defined (has a value)

② $\lim_{x \to c} f(x)$ exists

If any of 3 conditions aren't met, then we say $f(x)$ is discontinuous

(2) $\lim_{x \to c} f(x)$ exists

(3) $\lim_{x \to c} f(x) = f(c)$

then we say $f(x)$ is discontinuous at $x = c$.

Ex 1: Discuss continuity of $f(x) = \dfrac{2x}{x^2 - x}$.

When is $f(x)$ undefined? When denominator $= 0$.

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, 1 \quad \Rightarrow \text{ We have discontinuities @ } x = 0, 1$$

But what kind of discontinuity are they? Jump? Hole? VA?

Let's simplify the function

$$f(x) = \frac{2x}{x^2 - x} = \frac{2\cancel{x}}{\cancel{x}(x-1)} = \frac{2}{x-1}$$

What factor cancelled? $\underline{x} \Rightarrow \boxed{x = 0 \quad \text{Hole}}$

What factor remains (in denominator)? $\underline{x-1} \Rightarrow \boxed{x = 1 \quad \text{VA}}$

Ex 2: Discuss continuity of $f(x) = \dfrac{x^2 + 2x - 3}{x^2 + 5x - 6}$

When is $f(x)$ undefined? Denominator $= 0$.

$$x^2 + 5x - 6 = 0 \qquad\qquad \begin{matrix} -6 \\ \wedge \\ -1 + 6 = 5 \end{matrix}$$
$$x^2 - x + 6x - 6 = 0$$
$$x(x-1) + 6(x-1) = 0$$
$$(x+6)(x-1) = 0$$
$$x = -6, 1 \quad \Rightarrow \text{ Hence we have discontinuities @ } x = -6, 1$$

But what kind of discontinuity are they? Jump? ~~Hole~~? ~~VA~~?

But what kind of discontinuity are they? Jump? ~~Hole~~? ~~VA~~?

$$f(x) = \frac{x^2+2x-3}{(x+6)(x-1)}$$

Let's factor the numerator

$$x^2+2x-3$$
$$=(x+3)(x-1)$$

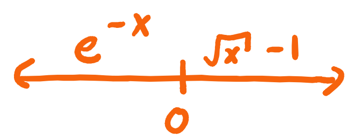
$$= \frac{(x+3)\cancel{(x-1)}}{(x+6)\cancel{(x-1)}}$$

What factor canceled? $\underline{x-1} \Rightarrow x=1$ Hole

What factor remains in denominator? $\underline{x+6} \Rightarrow x=-6$ VA

<u>Ex 3</u>: Discuss continuity of

$$f(x) = \begin{cases} e^{-x} & \text{if } x \le 0 \\ \sqrt{x}-1 & \text{if } x > 0 \end{cases}$$



To determine continuity, we need to check

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

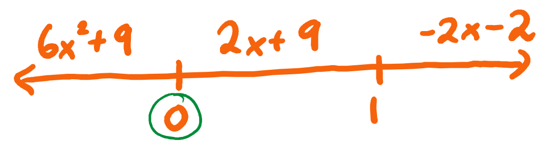
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-x} = e^{-0} = e^0 = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x}-1 = 0-1 = -1$$

$\neq \Rightarrow$ Discontinuity @ $x=0$ and moreso Jump @ $x=0$.

<u>Ex 4</u>: Discuss continuity of

$$f(x) = \begin{cases} 6x^2+9 & \text{if } x \le 0 \\ 2x+9 & \text{if } 0 < x < 1 \\ -2x-2 & \text{if } x \ge 1 \end{cases}$$



To determine continuity, we need to check

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

@ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 6x^2+9 = 6(0)^2+9 = 9$ ✓ $\Rightarrow$ No discontinuity @ $x=0$

ⓐ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 6x^2+9 = 6(0)^2+9 = 9$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x+9 = 2(0)+9 = 9$

$\checkmark \Rightarrow$ No discontinuity @ $x=0$

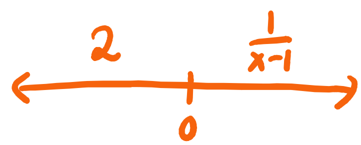
ⓑ $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2x+9) = 2(1)+9 = 11$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (-2x-2) = -2(1)-2 = -4$

$\neq \Rightarrow$ Jump @ $x=1$

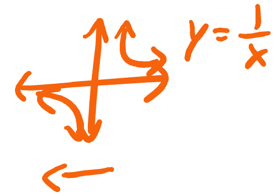
Ex 5: Discuss the continuity of

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x>0 \\ 2 & \text{if } x\le 0 \end{cases}$$



To determine continuity, we need to check

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$



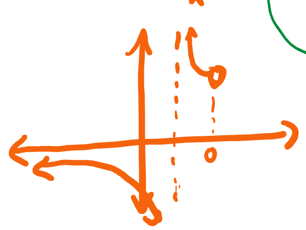
$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 2 = 2$

$\neq \Rightarrow$ Jump @ $x=0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1}{x-1} = \dfrac{1}{0-1} = \dfrac{1}{-1} = -1$

Check is $\dfrac{1}{x-1}$ when $x>0$ if it is continuous in that interval?

In addition VA @ $x=1$



$x \neq 1$ and $x>0 \Rightarrow 1$ lives in $x>0$

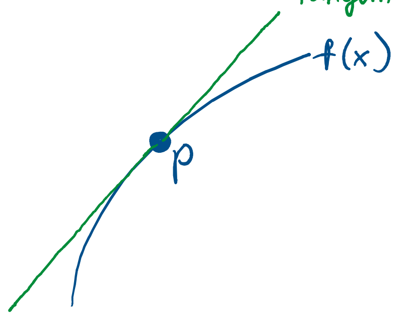
$\Rightarrow$ VA @ $x=1$

This VA won't matter if $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x>1 \\ 2 & \text{if } x\le 1 \end{cases}$

Lesson 5: The Derivative

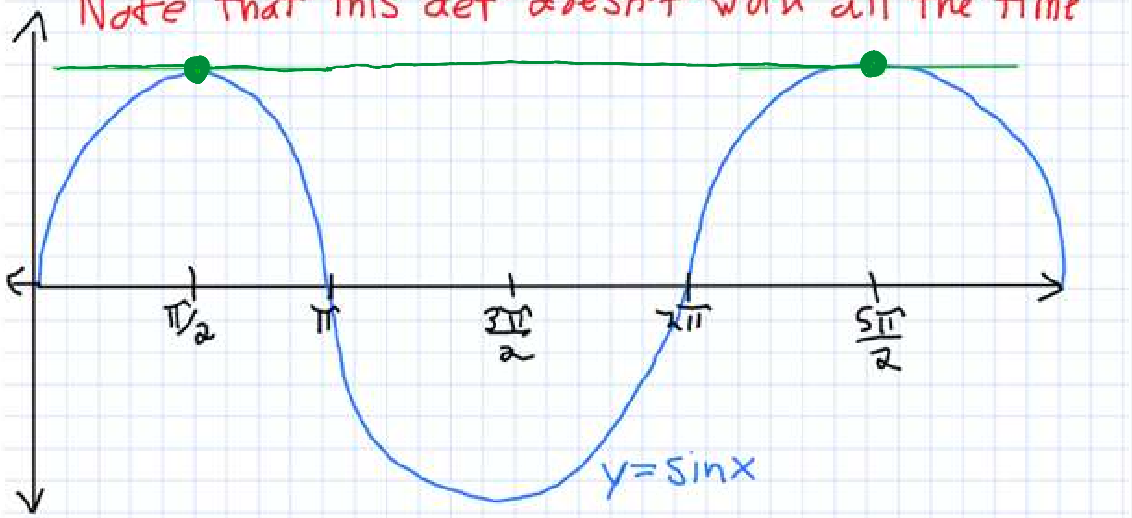
Tangent lines are Important in Calculus!!!

What is a Tangent line?



Tangent Line is Graph of f(x) at a point P that is straight line that touches f(x) at point P, but does not cross f(x).

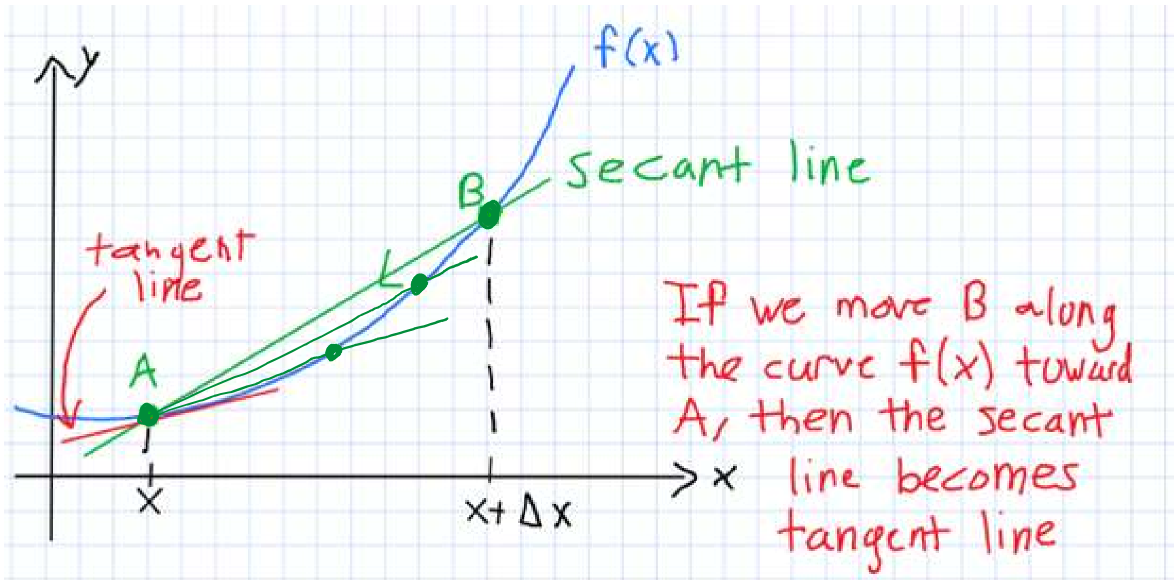
Note that this def doesn't work all the time

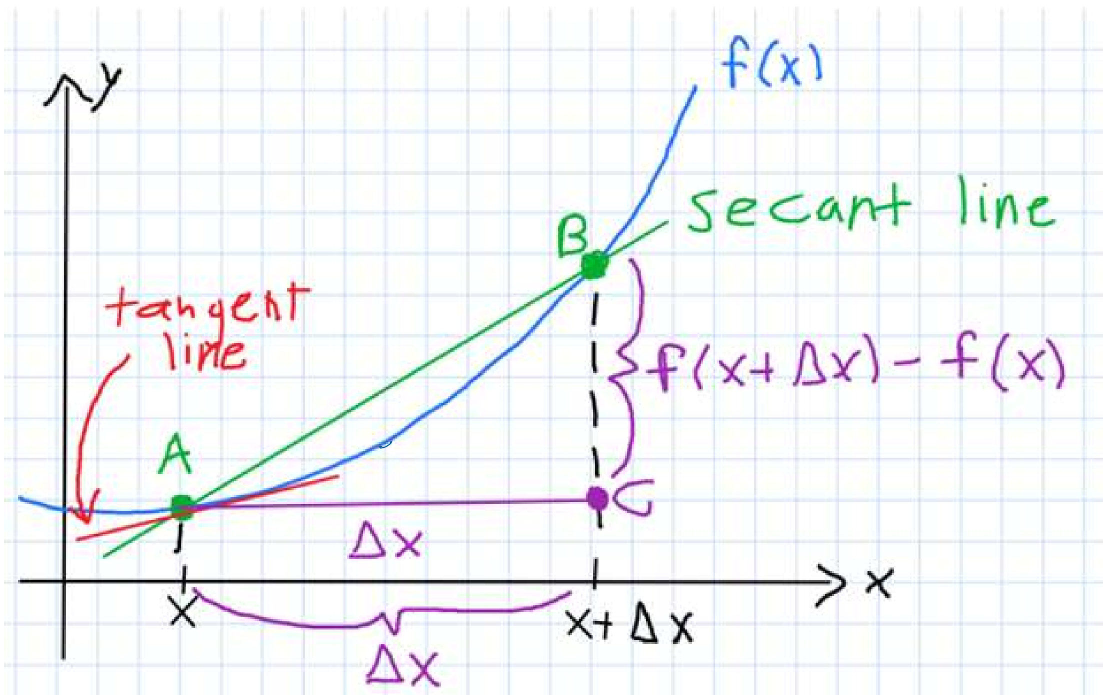


B/c look @ how those green lines cross

We need a precise definition for all scenarios. Before we do so, let's recap secant lines.

A secant line of f(x) is a straight line that goes through 2 distinct points on f(x).





If we want to find the tangent line to $f(x)$ at a specific point, we can do that with the secant.

How can we achieve that?

If we knew slope of tangent line, then we can use point-slope formula to find the eqn of tangent line.

Recall point-slope formula is

Given m-slope and (x_1, y_1)-point: $y - y_1 = m(x - x_1)$

Recall point-slope formula is
Given m-slope and (x_1, y_1)-point: $y - y_1 = m(x - x_1)$

$$\text{Slope of Tangent line} = \lim_{\Delta x \to 0} \text{Slope of secant line} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

also known as the difference quotient

How is the slope of the tangent line related to the derivative?
They are the same!

Def: The derivative of f(x) at x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{where } h = \Delta x$$

and provided that the limit exists.

Different Notations: y', $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}[f(x)]$

Game Plan: To find derivative using the limit definition of derivative, follow the following steps:

ex. $f(\Box) = \Box^2 + \Box$

① Find $f(x+h)$
② Find $-f(x)$
③ Find $f(x+h) - f(x)$ by adding ① + ②
④ Find $\frac{f(x+h) - f(x)}{h}$ by diving ③ by h
⑤ Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by taking $\lim_{h \to 0}$ of ④

Ex 1: Find the derivative of $f(x) = x + 5$ using the limit definition.

Step 1: $f(x+h)$
$f(x) = x + 5$
$f(x+h) = (x+h) + 5 = x + h + 5$

Step 2: $-f(x)$
$f(x) = x + 5$

Step 3: $f(x+h) - f(x)$

$$\frac{\cancel{x} + h + \cancel{5} + (-\cancel{x} - \cancel{5})}{h}$$

$f(x+h) - f(x) = h$

Step 2: $-f(x)$

$f(x) = x+5$

$-f(x) = -x-5$

$f(x+h) - f(x) = h$

Step 4: $\dfrac{f(x+h)-f(x)}{h} \overset{\text{by } ③}{=} \dfrac{h}{h} = 1$

Step 5: $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} \overset{\text{by } ④}{=} \lim\limits_{h\to 0} (1) = 1 = f'(x)$

Useful Formulas:

- $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- $a^2 - b^2 = (a-b)(a+b)$

Ex 2: Given $f(x) = x^2 - 3$.

ⓐ Find the slope of the tangent line. [i.e. $f'(x)$]

Step 1: $f(x+h) = (x+h)^2 - 3 = \cancel{x^2} + 2xh + h^2 - \cancel{3}$

Step 2: $-f(x) = -[x^2 - 3] = \cancel{-x^2} \qquad \cancel{+}3$

Step 3: $f(x+h) - f(x)$. Add ①+② $= 2xh + h^2$

Step 4: Divide ③ by h to get

$\dfrac{f(x+h)-f(x)}{h} = \dfrac{2xh+h^2}{h} = \dfrac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + h$

Step 5: Take the $\lim\limits_{h\to 0}$ of ④

$\lim\limits_{h\to 0} (2x+h) = 2x + 0 = 2x = f'(x)$

Game Plan: To find the eqn of the tangent line to $f(x)$ at the point $x=c$, follow the following steps:

① Find $f'(x)$.

② Calculate $f'(c)$

② Calculate $f'(c)$

③ Calculate $f(c)$

④ Plug ② + ③ into the point-slope formula

$$y - f(c) = m(x-c) \qquad m = f'(c) \text{ always}$$

$$y - f(c) = f'(c)(x-c)$$

Ex 2: Given $f(x) = x^2 - 3$.

ⓑ Find the equation of the tangent line to $f(x)$ at $x=2$.

Step 1: Find $f'(x)$.

By Part a, $f'(x) = 2x$

Step 2: Find $f'(2)$.

$f'(2) = 2(2) = 4$

Step 3: Find $f(2) = 2^2 - 3 = 4 - 3 = 1$

Step 4: Point-Slope Formula

$$y - f(2) = f'(2)(x-2)$$

$$y - 1 = 4(x-2)$$

$$y - 1 = 4x - 8$$

$$\underline{\quad +1 \qquad\quad +1 \quad}$$

$$y = 4x - 7$$

Ex 3: Determine what $f'(x)$ is given

$$\lim_{h\to 0} \frac{(2+h)^3 + (2+h)^2 - 12}{h} \text{ ?}$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Let $x = 2$:

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = f'(2)$$

$$f(2+h) = (2+h)^3 + (2+h)^2$$

Let $x = 2+h$

$f(2+h) = (2+h)^3 + (2+h)^2$

Let $x = 2+h$

$$f(x) = x^3 + x^2$$

Check that $f(2)$ is actually 12.

$$f(2) = 2^3 + 2^2 = 8 + 4 = 12$$

I know $f(x) = x^3 + x^2$. So $f'(x) = 3x^2 + 2x$

Why? In a future lesson I say why

Lesson 6 Basic Rules of Differentiation
Derivatives of Sine and Cosine
Derivatives of the Natural Exponential Functions

2-weeks from Tuesday is Exam 1 8pm-9pm

TI 30 Xa

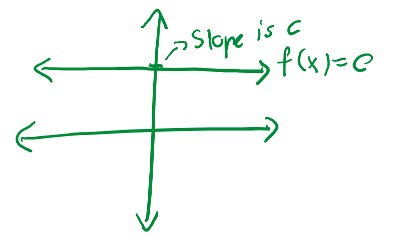
Sometime next week a calculator check

Basic Rules of Differentiation

① Constant Rule: For any constant c,

$$\frac{d}{dx}[c] = 0$$

Intuitively, this makes sense because the graph of $f(x)=c$ is a horizontal line.



Proof: Let's take derivative using the limit definition of derivatives.

Let $f(x) = c$

① $f(x+h) = c$

② $-f(x) = -c$

③ Add ①+② $= c - c = 0$

④ Divide by h: $\frac{0}{h} = 0$

⑤ $f'(x) = \lim_{h\to 0} 0 = 0 \Rightarrow \frac{d}{dx}[c] = 0$ Done!

② Power Rule: For any real number, n,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Ex 1: Find the derivative of $f(x) = x^2$

$n = 2$

$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x$$

Ex 2: Find the derivative of $f(x) = x^{-4}$

$n = -4$

Ex 2: Find the derivative of ...

$n = -4$

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

③ Constant Multiple Rule:

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

④+⑤ Sum/Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Ex 3: Find the derivative of $f(x) = x^5 + 5x^2$

$$f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}[x^5 + 5x^2]$$

$$= \frac{d}{dx}[x^5] + \frac{d}{dx}[5x^2] \quad \text{(by Rule 4)}$$

$$= \frac{d}{dx}[x^5] + 5\frac{d}{dx}[x^2] \quad \text{(by Rule 3)}$$

$$= 5x^{5-1} + 5 \cdot 2x^{2-1} \quad \text{(by Power Rule)}$$

$$= 5x^4 + 10x$$

Ex 4: Find the derivative of

$$\frac{1}{x^m} = x^{-m}$$

$$f(x) = \frac{3}{x^4} - 2x^2 + 6x + 7$$

We want to rewrite $f(x)$ with no fractions

$$f(x) = 3x^{-4} - 2x^2 + 6x + 7$$

$$f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}[3x^{-4} - 2x^2 + 6x + 7]$$

$$= \frac{d}{dx}[3x^{-4}] - \frac{d}{dx}[2x^2] + \frac{d}{dx}[6x] + \frac{d}{dx}[7] \quad \text{(by Rule 4+5)}$$

$$= 3\frac{d}{dx}[x^{-4}] - 2\frac{d}{dx}[x^2] + 6\frac{d}{dx}[x^1] + \boxed{\frac{d}{dx}[7]} \quad \text{(by Rule 3)}$$

$$= 3\frac{d}{dx}[x^{-4}] - 2\frac{d}{dx}[x^2] + 6\frac{d}{dx}[x^1] + \boxed{\frac{d}{dx}[7]} \text{ (by Rule 3)}$$

$$\frac{d}{dx}[7] = 0 \text{ (by Rule 1)}$$

$$= 3(-4)x^{-4-1} - 2(2)x^{2-1} + 6\cdot 1 x^{1-1} + 0$$

$$= -12x^{-5} - 4x^1 + 6x^0, \quad x^0 = 1$$

$$= -12x^{-5} - 4x + 6$$

$$= -\frac{12}{x^5} - 4x + 6$$

Ex 5: Find the derivative of $f(x) = \sqrt{x^1} + \sqrt[3]{x^2}$

$\sqrt[2]{x^1} = x^{1/2}$

$\sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$

Rewrite to have numerical powers.

$$f(x) = x^{1/2} + x^{2/3}$$

$$f'(x) = \frac{d}{dx}[x^{1/2} + x^{2/3}]$$

$$= \frac{d}{dx}[x^{1/2}] + \frac{d}{dx}[x^{2/3}]$$

$$= \frac{1}{2}x^{1/2 - 1} + \frac{2}{3}x^{2/3 - 1}$$

$$= \frac{1}{2}x^{-1/2} + \frac{2}{3}x^{-1/3}$$

Derivatives of Sine and Cosine

$$\frac{d}{dx}(\sin(x)) = \cos(x) \qquad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

Ex 6: Find y' of $y = 3\sin(x) - 4\cos(x)$

$$y' = \frac{d}{dx}[3\sin(x)] - \frac{d}{dx}[4\cos(x)]$$

$$= 3\underbrace{\frac{d}{dx}[\sin(x)]}_{\cos(x)} - 4\underbrace{\frac{d}{dx}[\cos(x)]}_{-\sin(x)}$$

$\cos(x)$ $\qquad -\sin(x)$

$= 3\cos(x) - 4(-\sin(x))$

$= 3\cos(x) + 4\sin(x)$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Ex 7: Find the x-value at which the derivative of $y = 10e^x$ is 1.

i.e. Solve $y' = 1$ for x.

$$y' = \frac{d}{dx}\left[10e^x\right] = 10\frac{d}{dx}\left[e^x\right] = 10e^x$$

I want $y' = 1$

$10e^x = 1$ ← solve for x.

$e^x = \frac{1}{10}$

$\ln(e^x) = \ln\left(\frac{1}{10}\right)$

$x = \ln\left(\frac{1}{10}\right)$

Ex 8 (Challenge): Find the derivative of

$$f(x) = 2x^2 + \frac{3}{x^3} - \sqrt[4]{x^5} + 2\sin(x) - 5\cos(x) + \pi e^x$$

Rewrite the function

$$f(x) = 2x^2 + 3x^{-3} - x^{5/4} + 2\sin(x) - 5\cos(x) + \pi e^x$$

$$f'(x) = 2\frac{d}{dx}\left[x^2\right] + 3\frac{d}{dx}\left[x^{-3}\right] - \frac{d}{dx}\left[x^{5/4}\right] + 2\frac{d}{dx}\left[\sin(x)\right] - 5\frac{d}{dx}\left[\cos(x)\right]$$

$$+ \pi\frac{d}{dx}\left[e^x\right]$$

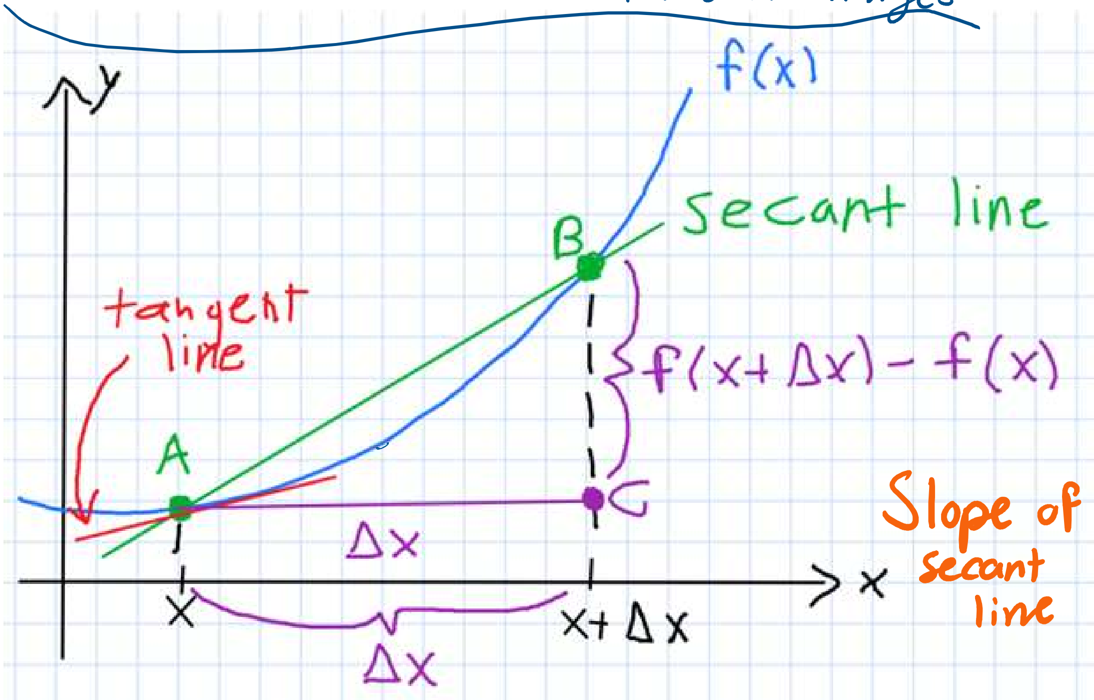
$$= 2\cdot 2x^{2-1} + 3(-3)x^{-3-1} - \frac{5}{4}x^{5/4-1} + 2\cos(x) - 5(-\sin(x)) + \pi e^x$$

$$= 2 \cdot 2x^{2-1} + 3(-3)x^{-3-1} - \frac{5}{4}x^{5/4-1} + 2\cos(x) - 5(-\sin(x)) + \pi e^x$$

$$= 4x^1 - 9x^{-4} - \frac{5}{4}x^{1/4} + 2\cos(x) + 5\sin(x) + \pi e^x$$

$$= 4x - \frac{9}{x^4} - \frac{5}{4}\sqrt[4]{x} + 2\cos(x) + 5\sin(x) + \pi e^x$$

Lesson 7: Instantaneous Rates of Changes



$$\text{Slope of secant line} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This quantity is also known as the average rate of change.

Average rate of change approaches a quantity is called instantaneous rate of change.

i.e. it's the derivative of f(x).

Ex 1: The initial population of a culture of bacteria is 1000. The population after t hours, P(t), is given by

$$P(t) = 2t^2 + 8t + 1000$$

ⓐ Find the number of bacteria present after 5 hrs.

$$P(5) = 2(5)^2 + 8(5) + 1000$$

$$P(5) = 2(5)^2 + 8(5) + 1000$$
$$= 2(25) + 40 + 1000$$
$$= 50 + 40 + 1000 = 1090$$

ⓑ Find the $\underbrace{\text{rate of change}}_{\text{i.e. is derivative}}$ of the population after 5 hrs.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

What's $P'(5)$?

$$P'(t) = \frac{d}{dt}\left[2t^2 + 8t + 1000\right] = 2 \cdot 2t^{2-1} + 8 + 0 = 4t + 8$$

$$P'(5) = 4(5) + 8 = 20 + 8 = 28$$

Position & Velocity Functions

- Position function $[s(t)]$ tells us how far away an object is
- Velocity function $[v(t)]$ tells us speed of an object with respect to direction.

To find velocity we take the derivative of the position.

$$v(t) = \frac{d}{dt}(s(t)) = \frac{ds}{dt} = s'(t)$$

Ex 2: An object is shot upward from the surface of ~~our~~ Earth. The position function is

$$s(t) = -4.9t^2 + 98t$$

ⓐ Find $v(t)$.

$$v(t) = s'(t) = \frac{d}{dt}\left[-4.9t^2 + 98t\right] = -4.9 \cdot 2t^{2-1} + 98$$
$$= -9.8t + 98$$

ⓑ Find $v(3)$

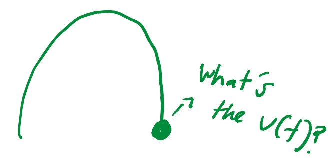
By ⓐ, $v(t) = -9.8t + 98$

$$v(3) = -9.8(3) + 98 = 68.6$$

$$v(3) = -9.8(3) + 98 = 68.6$$

(c) What is the velocity of the object when it hits the ground?

i.e. Solve $s(t)=0$ for t. Plug t into $v(t)$.



$$s(t) = 0$$

$$-4.9t^2 + 98t = 0$$

$$-4.9\,t(t-20) = 0$$

$$-4.9t = 0 \quad\Big|\quad t-20=0$$

$$t = 0 \quad\Big|\quad t = 20$$

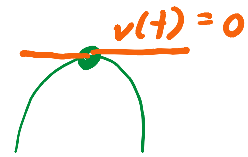
$$v(t) = -9.8t + 98$$

$$v(20) = -9.8(20) + 98$$

$$= -98$$

(d) When is the object at its highest point?

Solve $v(t)=0$ for t.



$$v(t) = -9.8t + 98 = 0$$

$$\frac{98}{9.8} = \frac{9.8t}{9.8}$$

$$10 = t$$

Ex 3: Let $C = 2\pi r$. What is the rate of change of C with respect to r?

i.e. Find $\frac{dc}{dr}$. $\quad \frac{dc}{dr} = \frac{d}{dr}[c]$

$$\frac{d}{dr}[c] = \frac{d}{dr}[2\pi r]$$

$$1 \cdot \frac{dc}{dr} = 2\pi \cdot 1 \cancel{\frac{dr}{dr}}^{\,1}$$

$$\frac{dc}{dr} = 2\pi$$

Ex 4: Let $p = 3q - 5$

(a) What is the $\underbrace{\text{rate of change of } p}_{\frac{d}{dq}}$ ...

(b) What is the rate of change of q with $\underbrace{\text{respect to } p}_{d}$?

with $\overbrace{\text{respect to } q}^{\frac{d}{dq}}$?

$$\frac{d}{dq}[p] = \frac{d}{dq}[3q-5]$$

$$1 \cdot \frac{dp}{dq} = 3\cancelto{1}{\frac{dq}{dq}}$$

$$\frac{dp}{dq} = 3$$

$\underbrace{\text{respect to } p}_{\frac{d}{dp}}$?

$$\frac{d}{dp}[p] = \frac{d}{dp}[3q-5]$$

$$1 \cdot \cancelto{1}{\frac{dp}{dp}} = 3 \cdot \frac{dq}{dp}$$

$$1 = 3\frac{dq}{dp}$$

$$\frac{1}{3} = \frac{dq}{dp}$$

Lesson 8: Product Rule

Product Rule says the derivative of $h(x) = u(x)\,v(x)$ is

$$\frac{d}{dx}\left[h(x)\right] = \frac{d}{dx}\left[u(x)v(x)\right]$$
$$= \frac{d}{dx}\left[u(x)\right]v(x) + u(x)\frac{d}{dx}\left[v(x)\right]$$
$$= u'(x)\,v(x) + u(x)\,v'(x)$$

Ex 1: Given $h(x) = 2x^3 e^x$. Compute $h'(x)$

Let $u(x) = 2x^3$ $\quad v(x) = e^x$
$\quad\;\; u'(x) = 6x^2$ $\quad v'(x) = e^x$

By product rule,
$$h'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= 6x^2 e^x + 2x^3 e^x$$
$$= (6x^2 + 2x^3)e^x$$

Ex 2: Given $h(x) = x^2 \sin(x)$. Compute $h'(\pi/6)$

Let $u(x) = x^2$ $\quad v(x) = \sin(x)$
$\quad\;\; u'(x) = 2x$ $\quad v'(x) = \cos(x)$

By product rule,
$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 2x\sin(x) + x^2\cos(x)$$

$$h'\left(\frac{\pi}{6}\right) = 2\frac{\pi}{6}\sin\left(\frac{\pi}{6}\right) + \left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\pi}{3}\cdot\frac{1}{2} + \frac{\pi^2}{36}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} \cdot \frac{1}{2} + \frac{\pi^2}{36} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} + \frac{\pi^2\sqrt{3}}{72}$$

Ex 3: Given $h(x) = \sqrt{x}\,(2x^2+4)$. Compute $h'(x)$.

Method 1: Use product rule

Let $u(x) = x^{1/2}$ $v(x) = 2x^2+4$

$u'(x) = \frac{1}{2}x^{-1/2}$ $v'(x) = 4x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= \frac{1}{2}x^{-1/2}(2x^2+4) + x^{1/2} \cdot 4x$$
$$= 1x^{3/2} + 2x^{-1/2} + 4x^{3/2}$$
$$= 5x^{3/2} + 2x^{-1/2}$$

Method 2: Expand $h(x)$

$$h(x) = x^{1/2}(2x^2+4)$$
$$= 2x^{5/2} + 4x^{1/2}$$

By Power Rule,

$$h'(x) = 2 \cdot \frac{5}{2} x^{3/2} + 4 \cdot \frac{1}{2} x^{-1/2}$$
$$= 5x^{3/2} + 2x^{-1/2}$$

Moral: Just b/c there is a product doesn't mean you need to use product rule.

Ex 4: Given $h(x) = (x^2+5x)(-3x^5+6)$. Find $h'(x)$.

Expand $h(x)$

	$-3x^5$	6
x^2	$-3x^7$	$6x^2$
$5x$	$-15x^6$	$30x$

$$h(x) = -3x^7 - 15x^6 + 6x^2 + 30x$$

By Power Rule,

$$h'(x) = -21x^6 - 90x^5 + 12x + 30$$

Ex 5: Given $h(x) = (x^3+x+1)(x^2+1)$. Find $h'(x)$.

	x^2	1
x^3	x^5	x^3
x	x^3	x
1	x^2	1

$$h(x) = x^5 + x^3 + x^2 + x^3 + x + 1$$
$$= x^5 + 2x^3 + x^2 + x + 1$$

By Power Rule,

$$h'(x) = 5x^4 + 6x^2 + 2x + 1$$

Ex 6: Find x-values at which $y = 4x^6 e^x$ has a horizontal tangent.

$$\left.\begin{array}{l}\text{Tangent} \Rightarrow f'(x) \\ \text{Horizontal} \longleftrightarrow \end{array}\right\} \left.\begin{array}{l} f'(x) \text{ is a slope} \\ \text{Horizontal lines} \\ \text{have slope} = 0 \end{array}\right\} f'(x) = 0$$

First find y'.

Let $u(x) = 4x^6$ $\quad v(x) = e^x$

$u'(x) = 24x^5$ $\quad v'(x) = e^x$

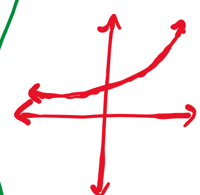
By Product Rule

$$y' = u'(x)v(x) + u(x)v'(x)$$
$$= 24x^5 e^x + 4x^6 e^x$$
$$= e^x(24x^5 + 4x^6)$$

Solve $y' = 0$.

$$e^x(24x^5 + 4x^6) = 0$$
$$e^x(4x^5)(6+x) = 0$$

$e^x = 0$	$4x^5 = 0$	$6 + x = 0$
	$x^5 = 0$	$x = -6$
	$x = 0$	



NEVER!

Ans: $x = -6, 0$

Ex 7: Let $h(x) = e^x \cos(x)$. Find $h'(x)$.

$u(x) = e^x$ $\quad v(x) = \cos(x)$

$u'(x) = e^x$ $\quad v'(x) = -\sin(x)$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= e^x \cos(x) - e^x \sin(x)$$
$$= e^x[\cos(x) - \sin(x)]$$

Ex 8: Let $h(x) = (x^2 + 2x + 1)(\cos x - \sin x + e^x)$. Compute

Let $u(x) = x^2 + 2x + 1$ $\quad v(x) = \cos(x) - \sin(x) + e^x$

$u'(x) = 2x + 2$ $\quad v'(x) = -\sin(x) - \cos(x) + e^x$

Let $u(x)$

$u'(x) = 2x + 2 \qquad v'(x) = -\sin(x) - \cos(x) + e^x$

By product rule,

$$h'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$
$$= (2x+2)(\cos x - \sin x + e^x) + (x^2 + 2x + 1)(-\sin(x) - \cos(x) + e^x)$$

Monday, February 2, 2026 1:16 PM

Calculator Check on 2/4 } 12 questions
TI 30Xa

Multiple Choice
Lessons 2-10
8pm-9pm Tuesday 2/10

Lesson 9: Quotient Rule; Derivatives of other Trig Functions

Quotient Rule:

Quotient Rule says the derivative of $h(x) = \dfrac{u(x)}{v(x)}$ is

$$\frac{d}{dx}\left[h(x)\right] = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

Ex 1: Let $h(x) = \dfrac{1}{x^2}$. Find $h'(x)$

Method 1: Power Rule

Rewrite $h(x)$

$$h(x) = x^{-2}$$
$$h'(x) = -2x^{-2-1}$$
$$= -2x^{-3}$$
$$= -\frac{2}{x^3}$$

Method 2: Quotient Rule

Let $u(x) = 1$ $\quad v(x) = x^2$

$u'(x) = 0$ $\quad v'(x) = 2x$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{0 \cdot x^2 - 1 \cdot (2x)}{(x^2)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

Ex 2: Let $h(x) = \dfrac{x^2+1}{x^3-3x}$. Find $h'(x)$.

Let $u(x) = x^2+1$ $\quad v(x) = x^3 - 3x$

$u'(x) = 2x$ $\quad v'(x) = 3x^2 - 3$

By quotient rule,

$h'(x) = u'(x)v(x) - u(x)v'(x)$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{2x(x^3-3x) - \overbrace{(x^2+1)(3x^2-3)}}{(x^3-3x)^2}$$

	x^2	1
$3x^2$	$3x^4$	$\cancel{3x^2}$
-3	$\cancel{-3x^2}$	-3

$$= \frac{2x(x^3-3x) - (3x^4-3)}{(x^3-3x)^2}$$

$$= \frac{2x^4-6x^2-3x^4+3}{(x^3-3x)^2}$$

$$= \frac{-x^4-6x^2+3}{(x^3-3x)^2}$$

Ex 3: Let $h(x) = \frac{\sin(x)}{x+\sin(x)}$. Find $h'(x)$.

Let $u(x) = \sin(x)$ $\qquad v(x) = x+\sin(x)$

$u'(x) = \cos(x)$ $\qquad v'(x) = 1+\cos(x)$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{\cos(x)\left[x+\sin(x)\right] - \sin(x)\left[1+\cos(x)\right]}{(x+\sin(x))^2}$$

$$= \frac{x\cos(x) + \cancel{\cos(x)\sin(x)} - \sin(x) - \cancel{\sin(x)\cos(x)}}{(x+\sin(x))^2}$$

$$= \frac{x\cos(x) - \sin(x)}{(x+\sin(x))^2}$$

$$= \frac{x\cos(x) - \sin(x)}{(x+\sin(x))^2}$$

Ex 4: Let $h(x) = \dfrac{x^2+2x+\pi}{e^x}$ Find $h'(x)$

Let $u(x) = x^2+2x+\pi \qquad v(x) = e^x$

$u'(x) = 2x+2 \qquad v'(x) = e^x$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{(2x+2)e^x - (x^2+2x+\pi)e^x}{(e^x)^2}$$

$$= \frac{(\cancel{2x}+2-x^2-\cancel{2x}-\pi)\cancel{e^x}}{(e^x)^{\cancel{2}}}$$

$$= \frac{(-x^2+2-\pi)}{e^x}$$

Derivatives of Other Trig Functions

$\tan(x) = \dfrac{\sin(x)}{\cos(x)} \qquad \cot(x) = \dfrac{\cos(x)}{\sin(x)}$

$\sec(x) = \dfrac{1}{\cos(x)} \qquad \csc(x) = \dfrac{1}{\sin(x)}$

Ex 5: Let $h(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Find $h'(x)$

Let $u(x) = \sin(x) \qquad v(x) = \cos(x)$

$u'(x) = \cos(x) \qquad v'(x) = -\sin(x)$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$\boxed{\cos^2(x) + \sin^2(x) = 1 \quad \text{Always}}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x) \quad \Rightarrow \frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\left.\begin{aligned} \frac{d}{dx}[\cot(x)] &= -\csc^2(x) \\ \frac{d}{dx}[\sec(x)] &= \sec(x)\tan(x) \\ \frac{d}{dx}[\csc(x)] &= -\csc(x)\cot(x) \end{aligned}\right\} \text{Can be derived using quotient rule}$$

Ex 6: Given $h(x) = 3\sin(x)\tan(x)$. Compute $h'(x)$.

Let $u(x) = 3\sin(x)$ $\qquad v(x) = \tan(x)$

$u'(x) = 3\cos(x)$ $\qquad v'(x) = \sec^2(x)$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$

$$= 3\cos(x)\tan(x) + 3\sin(x)\sec^2(x)$$

$$= 3\cancel{\cos(x)}\frac{\sin(x)}{\cancel{\cos(x)}} + 3\sin(x)\sec^2(x)$$

$$= 3\sin(x) + 3\sin(x)\sec^2(x)$$

Ex 7: Given $h(x) = \dfrac{\tan(x)}{e^x + \sec(x)}$. Find $h'(x)$.

Let $u(x) = \tan(x)$ $\qquad v(x) = e^x + \sec(x)$

$e^{\wedge} + \sec(x)$

Let $u(x) = \tan(x)$ $\quad v(x) = e^x + \sec(x)$

$u'(x) = \sec^2(x)$ $\quad v'(x) = e^x + \sec(x)\tan(x)$

By quotient rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{\sec^2(x)\left[e^x + \sec(x)\right] - \tan(x)\left[e^x + \sec(x)\tan(x)\right]}{(e^x + \sec(x))^2}$$

$$= \frac{e^x\sec^2(x) + \sec^3(x) - \tan(x)e^x - \sec(x)\tan^2(x)}{(e^x + \sec(x))^2}$$

Lesson 10: The Chain Rule

Recall composition of functions

Let $y = f(g(x))$ ⇒ g - inner function
f - outer function

Ex 1: Determine f and g for $y = (3x+1)^2$

$f(x) = (x)^2$ $g(x) = 3x+1$

Check $y = f(g(x)) = f(3x+1) = (3x+1)^2$ ✓

Ex 2: Determine f and g for $y = \sin^2 x = (\sin x)^2$

$f(x) = x^2$ $g(x) = \sin(x)$

Check $y = f(g(x)) = f(\sin x) = (\sin x)^2$ ✓

Ex 3: Determine f and g for $y = \tan(3x)$

$f(x) = \tan(x)$ $g(x) = 3x$

Ex 4: Determine f and g for $y = \sqrt[3]{2x+1}$

$f(x) = \sqrt[3]{x}$ $g(x) = 2x+1$

Chain Rule

Let $y = f(g(x))$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

Ex 1: Find y' of $y=(3x+1)^2$.

Method 1: Expand and then use power rule

Recall $(a+b)^2 = a^2+2ab+b^2$

So $y=(3x)^2+2(3x)(1)+1^2$

$= 9x^2+6x+1$

$y' = 9(2)x^1+6$

$= 18x+6$

Method 2: Chain Rule

$f(x) = x^2$ $\qquad$ $g(x) = 3x+1$

$f'(x) = 2x$ $\qquad$ $g'(x) = 3$

By chain rule,

$y' = f'(g(x)) \cdot g'(x)$

$= f'(3x+1) \cdot 3$

$= 2(3x+1) \cdot 3$

$= 6(3x+1)$

$= 18x+6$

Ex 2: Find y' of $y=(3x^2+2x+1)^7$

Let $f(x) = x^7$ $\qquad$ $g(x) = 3x^2+2x+1$

$f'(x) = 7x^6$ $\qquad$ $g'(x) = 6x+2$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(3x^2 + 2x + 1) \cdot (6x + 2)$$
$$= 7(3x^2 + 2x + 1)^6 (6x + 2)$$

Ex 3: Find y' of $y = 2\cos^3 x$.
$$= 2(\cos x)^3$$

Let $f(x) = 2x^3$ $\quad$ $g(x) = \cos x$
$f'(x) = 6x^2$ $\quad$ $g'(x) = -\sin x$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(\cos(x)) \cdot (-\sin x)$$
$$= 6(\cos x)^2 (-\sin x)$$
$$= -6\cos^2 x \sin x$$

Ex 4: Find y' of $y = 5\tan(e^{3x})$

Let $f(x) = 5\tan(x)$ $\quad$ $g(x) = e^{3x}$ ← To find this I need another chain rule.

$h(x) = e^x$ $\quad$ $j(x) = 3x$
$h'(x) = e^x$ $\quad$ $j'(x) = 3$

$f'(x) = 5\sec^2(x)$ $\quad$ $g'(x) = h'(j(x)) \cdot j'(x)$
$$= h'(3x) \cdot 3$$
$$= e^{3x} \cdot 3$$

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(e^{3x}) \cdot 3e^{3x}$$

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(e^{3x}) \cdot 3e^{3x}$$
$$= 5\sec^2(e^{3x}) \cdot 3e^{3x}$$
$$= 15e^{3x}\sec^2(e^{3x})$$

Ex 5: Find y' of $y = \left(\frac{2x}{3x^2+x}\right)^3$

Rewrite: $y = \left(\frac{2\cancel{x}}{\cancel{x}(3x+1)}\right)^3$
$$= \left(\frac{2}{3x+1}\right)^3$$
$$= \frac{2^3}{(3x+1)^3}$$
$$= \frac{8}{(3x+1)^3}$$
$$= 8(3x+1)^{-3}$$

Let $f(x) = 8x^{-3}$ $g(x) = 3x+1$
$f'(x) = 8(-3)x^{-4}$ $g'(x) = 3$
$= -24x^{-4}$

By chain rule,
$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(3x+1) \cdot 3$$
$$= -24(3x+1)^{-4} \cdot 3$$
$$= \frac{-72}{(3x+1)^4}$$

Ex 6: Given $y = \frac{15}{\sqrt[3]{x^2+1}}$. Find $y'(1)$.

Rewrite so no quotient rule.
$$y = \frac{15}{(x^2+1)^{1/3}} = 15(x^2+1)^{-1/3}$$

Let $f(x) = 15x^{-1/3}$ $g(x) = x^2+1$
$f'(x) = 15\left(\frac{-1}{3}\right)x^{-1/3-1}$ $g'(x) = 2x$
$= -5x^{-4/3}$

By chain rule,

By chain rule,

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(x^2+1) \cdot 2x$$
$$= -5(x^2+1)^{-4/3}(2x)$$
$$= \frac{-5 \cdot 2x}{(x^2+1)^{4/3}} = \frac{-10x}{(x^2+1)^{4/3}}$$

I want $y'(1)$

$$y'(1) = \frac{-10(1)}{(1^2+1)^{4/3}} = \frac{-10}{2^{4/3}} = \frac{-10}{2^{3/3}\,2^{1/3}} = \frac{-10}{2 \cdot \sqrt[3]{2}} = \frac{-5}{\sqrt[3]{2}}$$

Lesson 11: The Chain Rule Pt 2
Derivative of Natural Logarithmic Function

Recall when $y=f(g(x))$ then $y'=f'(g(x))\cdot g'(x)$

Trig & Exponential Functions w/ chain Rule

Note $\sin^2 x = (\sin x)^2$ ← Check by plugging $x=\frac{\pi}{2}$ into both. Are they equal?

$\neq$

$\sin(x^2)$

Ex 1: Find y' when $y=\sin(x^2)$

Let $f(x)=\sin(x)$ $\quad g(x)=x^2$

$f'(x)=\cos(x)$ $\quad g'(x)=2x$

By chain rule,

$$\begin{aligned} y' &= f'(g(x))\cdot g'(x) \\ &= f'(x^2)\cdot 2x \\ &= \cos(x^2)\cdot 2x \\ &= 2x\cos(x^2) \end{aligned}$$

Ex 2: Find y' when $y=\sec(-2x+1)\tan(3x)$

Product Rule: $u(x)=\sec(-2x+1)$ $\quad v(x)=\tan(3x)$

↳ Chain Rule $\quad$ ↳ Chain Rule

$u'(x)=\sec(-2x+1)\tan(-2x+1)(-2)$ $\quad v'(x)=\sec^2(3x)\cdot 3$

By product rule,

By product rule,

$$y' = u'v + uv'$$
$$= \sec(-2x+1)\tan(-2x+1)(-2)\tan(3x)$$
$$+ \sec(-2x+1)\sec^2(3x)\cdot 3$$
$$= -2\sec(-2x+1)\tan(-2x+1)\tan(3x)$$
$$+ 3\sec(-2x+1)\sec^2(3x)$$

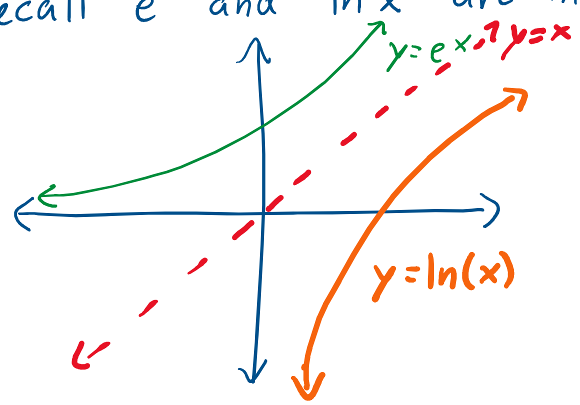
Ex 3: Find y' when $y = e^{(1-2x)^4}$

By Chain Rule,

$$y' = e^{(1-2x)^4}\frac{d}{dx}\left[(1-2x)^4\right]$$
$$= e^{(1-2x)^4}\cdot 4(1-2x)^3\frac{d}{dx}\left[1-2x\right]$$
$$= e^{(1-2x)^4}\cdot 4(1-2x)^3(-2)$$
$$= -8(1-2x)^3 e^{(1-2x)^4}$$

Derivative of Logarithmic Functions

Recall e^x and $\ln x$ are inverses. B/c they are inverses



$$\text{Slope of } \ln(x) = \frac{1}{\text{Slope of } e^x}$$

So for $x > 0$,

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

Ex 4: Find y' when $y = x\ln(x)$

Product Rule: $u = x$ $\quad v = \ln(x)$

$u' = 1$ $\quad v' = \frac{1}{x}$

So $y' = u'v + uv'$

$= 1 \cdot \ln(x) + x \cdot \frac{1}{x}$

$= \ln(x) + 1$

Ex 5: Find y' when $y = \ln(3x^2 + x + 1)$

Chain Rule

$$y' = \frac{1}{3x^2+x+1} \frac{d}{dx}(3x^2+x+1)$$

$$= \frac{1}{3x^2+x+1} \cdot (6x+1)$$

$$= \frac{6x+1}{3x^2+x+1}$$

Recall

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$\ln(xy) = \ln(x) + \ln(y)$

$\ln(x^m) = m\ln(x)$

$\ln(1) = 0$

$\ln(e^x) = x$

Ex 6: Find y' when $y = \ln\sqrt[3]{\frac{x^2+1}{2x-1}}$

Rewrite y with Logarithmic Rules to avoid double chain w/ quotient rule

Rewrite y with Logarithmic Rules to avoid double chain w/ quotient rule

$$y = \ln\left[\left(\frac{x^2+1}{2x-1}\right)^{1/3}\right] = \frac{1}{3}\ln\left(\frac{x^2+1}{2x-1}\right) = \frac{1}{3}\left[\ln(x^2+1) - \ln(2x-1)\right]$$
$$= \frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(2x-1)$$

By chain rule,

$$y' = \frac{1}{3} \cdot \frac{1}{x^2+1} \frac{d}{dx}(x^2+1) - \frac{1}{3} \cdot \frac{1}{2x-1} \cdot \frac{d}{dx}(2x-1)$$
$$= \frac{1}{3} \cdot \frac{1}{x^2+1} \cdot 2x - \frac{1}{3} \cdot \frac{1}{2x-1} \cdot 2$$
$$= \frac{2x}{3(x^2+1)} - \frac{2}{3(2x-1)}$$

Ex 6: Find y' when $y = \ln(\sin(x) \cdot e^{2x})$

Rewrite to avoid chain rule w/ product rule.

$$y = \ln(\sin(x)) + \ln(e^{2x})$$
$$= \ln(\sin(x)) + 2x$$

By chain rule,

$$y' = \frac{1}{\sin(x)} \frac{d}{dx}(\sin(x)) + 2$$
$$= \frac{1}{\sin(x)} \cdot \cos(x) + 2$$
$$= \frac{\cos(x)}{\sin(x)} + 2$$
$$= \cot(x) + 2$$

<u>Ex 7</u>: Find y' of $y = \ln\left(\frac{x^{3/2}(x+1)}{(x+3)^2}\right)$

Rewrite

$$y = \ln\left(x^{3/2}(x+1)\right) - \ln\left[(x+3)^2\right]$$

$$= \ln\left(x^{3/2}\right) + \ln(x+1) - \ln\left[(x+3)^2\right]$$

$$= \frac{3}{2}\ln(x) + \ln(x+1) - 2\ln(x+3)$$

By chain rule,

$$y' = \frac{3}{2}\cdot\frac{1}{x} + \frac{1}{x+1}\cdot\cancelto{1}{\frac{d}{dx}(x+1)} - \frac{2}{x+3}\cancelto{1}{\frac{d}{dx}(x+3)}$$

$$= \frac{3}{2}\cdot\frac{1}{x} + \frac{1}{x+1} - \frac{2}{x+3}$$

Question 9 of 54

For

$$f(x) = \begin{cases} x^2 - 3, & x \le -2 \\ -4x + 1, & x > -2 \end{cases}$$

Choose the correct statement(s) below.

I. $\lim_{x \to -2} f(x)$ does not exist.

II. $f(x)$ is continuous at $x = -2$.

III. $\lim_{x \to -2^-} f(x) = 1$

- Only II is true
- Only III is true
- Only I and II are true
- Only I and III are true
- Only II and III are true
- Only I is true

$x^2 - 3$ | $-4x+1$

-2

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = f(-2)$

continuous

$\lim_{x \to -2^-} (x^2 - 3) = (-2)^2 - 3 = 4 - 3 = 1$ ≠ DNE

$\lim_{x \to -2^+} (-4x+1) = (-4)(-2) + 1 = 8 + 1 = 9$

I ✓ ~~II~~ III ✓ $\boxed{D}$

Question 14 of 54

Find $f'(2\pi)$ if $f(x) = (x^2 + 4)(x + \sin x)$.

- $16\pi^2 + 4$
- 8π
- $16\pi^2 + 8$
- $12\pi^2 + 2\pi + 4$
- $8\pi^3 + 8\pi$
- -16π

$f(x) = (x^2 + 4)(x + \sin x)$

Product Rule,

$u = x^2 + 4$ $\quad$ $v = x + \sin x$

$u' = 2x$ $\quad$ $v' = 1 + \cos x$

By product rule,

$f' = u'v + uv'$

$= 2x(x + \sin x) + (x^2 + 4)(1 + \cos x)$

$f'(2\pi) = 2(2\pi)(2\pi + \cancel{\sin(2\pi)}^{0})$

$+ ((2\pi)^2 + 4)(1 + \cancel{\cos(2\pi)}^{1})$

$= 4\pi(2\pi) + (4\pi^2 + 4)(2)$

$= 8\pi^2 + 8\pi^2 + 8$

$= 16\pi^2 + 8$

C

Question 18 of 54

The derivative of a function $f(x)$ is found by computing

$$f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{(x+h)^2-1} - \sqrt[3]{x^2-1}}{h}.$$

Which of the following could be $f(x)$?

- ○ $f(x) = \sqrt[3]{(x+h)^2-1}$
- ○ $f(x) = \sqrt[3]{x^2}$
- ○ $f(x) = \dfrac{1}{\sqrt[3]{(x+h)^2-1}}$
- ○ $f(x) = \sqrt[3]{(x+h)^2}$
- ○ $f(x) = \dfrac{1}{\sqrt[3]{x^2-1}}$
- ○ $f(x) = \sqrt[3]{x^2-1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt[3]{(x+h)^2-1} - \sqrt[3]{x^2-1}}{h}$$

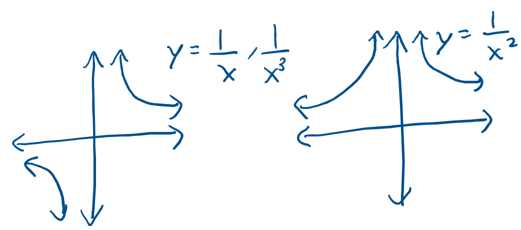
Let's guess $f(x) = \sqrt[3]{x^2-1}$ F

Check $f(x+h) \stackrel{?}{=} \sqrt[3]{(x+h)^2-1}$ ✓

Question 19 of 54

Which of the following is **NOT** equal to ∞?

- ○ $\lim\limits_{x\to 2^+} \dfrac{1}{x-2}$
- ○ $\lim\limits_{x\to 4^-} \dfrac{1}{(4-x)^2}$
- ○ $\lim\limits_{x\to 8^-} \dfrac{2}{(x-8)^3}$
- ○ $\lim\limits_{x\to 4^+} \dfrac{1}{(4-x)^2}$
- ○ $\lim\limits_{x\to 8^+} \dfrac{2}{(x-8)^3}$
- ○ $\lim\limits_{x\to 3^+} \dfrac{1}{\sqrt{x-3}}$



~~A~~ $\lim\limits_{x\to 2^+} \dfrac{1}{x-2} = \infty$

~~B~~ $\lim\limits_{x\to 4^-} \dfrac{1}{(4-x)^2} = \lim\limits_{x\to 4^-} \dfrac{1}{(x-4)^2} = \infty$

$\boxed{C}$ $\lim\limits_{x\to 8^-} \dfrac{2}{(x-8)^3} = -\infty$

Question 20 of 54

Find the x value at which $g(x) = xe^x$ has a horizontal tangent line.

- ○ $\frac{1}{2}$
- ○ -2
- ○ 2
- ○ -1
- ○ 0
- ○ 1

$g'(x) = 0$

$g(x) = xe^x$

$u = x \qquad v = e^x$

$u' = 1 \qquad v' = e^x$

$g'(x) = e^x + xe^x = 0$

$e^x(1+x) = 0$

$e^x = 0 \qquad 1 + x = 0$

NEVER $\qquad -1 = x$

Question 22 of 54

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Which of the following functions has a hole at $x = 4$?

- $f(x) = \frac{1}{x-4}$
- $f(x) = \frac{x^2-16}{x-4}$
- $f(x) = \frac{x+4}{x^2-16}$
- $f(x) = \frac{x+4}{x-4}$
- $f(x) = x - 4$
- $f(x) = \frac{x}{4-x}$

HOLE => Factors to cancel. → Hole @ $x=4$, $x-4=0$ byebye

~~A~~ B C ~~D~~ ~~E~~ ~~F~~

B) $f(x) = \frac{x^2-16}{x-4} = \frac{(\cancel{x-4})(x+4)}{\cancel{x-4}} = x+4$

c) $f(x) = \frac{x+4}{x^2-16} = \frac{\cancel{x+4}}{(x-4)(\cancel{x+4})} = \frac{1}{x-4}$

Question 23 of 54

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Given $y = \left(\frac{x}{3} + \frac{6}{x}\right)^3$, find $y'(-3)$.

- $\frac{1}{3}$
- -9
- -27
- $-\frac{2}{3}$
- 0
- 3

$y = \left(\frac{x}{3} + \frac{6}{x}\right)^3$

$y = \left(\frac{x}{3} + 6x^{-1}\right)^3$

$y' = 3\left(\frac{x}{3} + 6x^{-1}\right)^2 \frac{d}{dx}\left(\frac{x}{3} + 6x^{-1}\right)$

$= 3\left(\frac{x}{3} + \frac{6}{x}\right)^2 \cdot \left(\frac{1}{3} - 6x^{-2}\right)$

$= 3\left(\frac{x}{3} + \frac{6}{x}\right)^2 \left(\frac{1}{3} - \frac{6}{x^2}\right)$

$y'(-3) = 3\left(\frac{-3}{3} + \frac{6}{-3}\right)^2\left(\frac{1}{3} - \frac{6}{(-3)^2}\right)$

$= 3(-1-2)^2\left(\frac{1}{3} - \frac{6}{9}\right)$

$= 3(-3)^2\left(\frac{1}{3} - \frac{2}{3}\right)$

$= 3 \cdot 9 \left(\frac{-1}{3}\right) = -9$

Question 32 of 54

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Which of the following is equal to $\cos x$?

- $\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$
- $-\sin x$

$\cos(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$\parallel$

$f'(x)$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

- $\lim_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$
- $-\sin x$
- $\lim_{x\to 0} \dfrac{\cos(x+h)-\cos(x)}{h}$
- $\lim_{h\to 0} \dfrac{\sin(x+h)+\sin x}{h}$
- $\lim_{x\to 0} \dfrac{\sin(x+h)-\sin x}{h}$
- $\lim_{h\to 0} \dfrac{\cos(x+h)-\cos x}{h}$

$f(x)=\sin(x)$

$f'(x)=\cos(x)$

A $\quad = \lim_{h\to 0} \dfrac{\sin(x+h)-\sin(x)}{h}$

Question 35 of 54

Find $y'(x)$.

$$y=\frac{3}{(4-x^2)^2}$$

- $\dfrac{-12x}{4-x^2}$
- $\dfrac{-6}{4-x^2}$
- $\dfrac{-6}{(4-x^2)^3}$
- $\dfrac{12x}{4-x^2}$
- $\dfrac{12x}{(4-x^2)^3}$
- $\dfrac{-12x}{(4-x^2)^3}$

$y=\dfrac{3}{(4-x^2)^2}$

Rewrite $\quad y=3(4-x^2)^{-2}$

Chain Rule

$y=3(-2)(4-x^2)^{-3}\dfrac{d}{dx}(4-x^2)$

$=-6(4-x^2)^{-3}(-2x)$

$=12x(4-x^2)^{-3}$

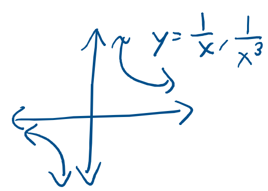
$=\dfrac{12x}{(4-x^2)^3}$ E

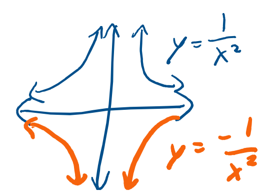
Question 36 of 54

Which of the following statements is **TRUE**:

- $\lim_{x\to 0} \dfrac{5}{x}=5$
- $\lim_{x\to 8} \dfrac{1}{x-8}=\infty$
- $\lim_{x\to -3} \dfrac{-1}{x+3}=+\infty$
- $\lim_{x\to 1} \dfrac{-7}{(x-1)^3}=-\infty$
- $\lim_{x\to 0} \dfrac{1}{x^2}=0$
- $\lim_{x\to \frac{1}{2}} \dfrac{-3}{(2x-1)^2}=-\infty$

~~A~~ ~~B~~ ~~C~~ ~~D~~ ~~E~~ F





Question 39 of 54

Choose the number of correct statements regarding the piecewise function below.

$$f(x) = \begin{cases} 6x^2 + 9 & x \le 0 \\ 13x + 9 & 0 < x < 1 \\ -13x + 9 & x \ge 1 \end{cases}$$

I. $\lim_{x\to 0} f(x) = 9.$

II. $\lim_{x\to 1^-} f(x) = 22.$

III. $\lim_{x\to 1^+} f(x) = -4.$

IV. $f(x)$ has a hole at $x = 0$.

V. $f(x)$ has a jump at $x = 1$.

- ○ Only one statement is true.
- ◉ None of the statements is true.
- ○ All five statements are true.
- ○ Only four statements are true.
- ○ Only two statements are true.
- ○ Only three statements are true.

$6x^2+9$ $\quad 13x+9 \quad$ $-13x+9$

$0 \qquad 1$

✓ I $\lim_{x\to 0^-} (6x^2+9) \stackrel{?}{=} \lim_{x\to 0^+} (13x+9)$

$9 = 9$ ✓

✓ II $\lim_{x\to 1^-} (13x+9) = 13+9 = 22$

✓ III $\lim_{x\to 1^+} (-13x+9) = -13+9 = -4$

~~IV~~ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0)$

$9 = 9 \stackrel{?}{=} f(0)$

V $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ ✓

By II & III

0

Lesson 12: Higher Order Derivatives

The derivative of a function, $f(x)$, is also called first derivative,

$$y',\ f'(x),\ \frac{dy}{dx},\ \frac{d}{dx}[f(x)]$$

If we take the derivative of the first derivative of $y=f(x)$, then we get second derivative

$$y'',\ f''(x),\ \frac{d^2y}{dx^2},\ \frac{d^2}{dx^2}[f(x)]$$

Take the derivative n times, we get the nth derivative.

$$y^{(n)},\ f^{(n)}(x),\ \frac{d^ny}{dx^n},\ \frac{d^n}{dx^n}[f(x)]$$

Ex 1: Find the first three derivatives of

$$R(x) = 3x^2 + 8x^{1/2} + e^x$$

① $R'(x) = 3(2)x + 8\left(\frac{1}{2}\right)x^{-1/2} + e^x$

$= 6x + 4x^{-1/2} + e^x$

② $R''(x) = \frac{d}{dx}\left[6x + 4x^{-1/2} + e^x\right]$

$= 6 + 4\left(-\frac{1}{2}\right)x^{-3/2} + e^x$

$= 6 - 2x^{-3/2} + e^x$

③ $R'''(x) = \frac{d}{dx}\left[6 - 2x^{-3/2} + e^x\right]$

$= 0 - 2\left(\frac{-3}{2}\right)x^{-5/2} + e^x$

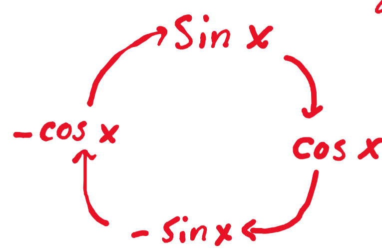
$= 3x^{-5/2} + e^x$

Ex 2: Find the first five derivatives of $y = \sin(x)$.

Recall $\frac{d}{dx}[\sin x] = \cos x$ $\quad \frac{d}{dx}[\cos x] = -\sin x$

Derivatives of $\sin x$ & $\cos x$

Derivatives of $\sin x$ & $\cos x$



① $y' = \frac{d}{dx}[\sin x] = \cos(x)$

② $y'' = \frac{d}{dx}[\cos(x)] = -\sin(x)$

③ $y''' = \frac{d}{dx}[-\sin x] = -\frac{d}{dx}[\sin(x)] = -\cos(x)$

④ $y^{(4)} = \frac{d}{dx}[-\cos(x)] = -\frac{d}{dx}[\cos(x)] = -(-\sin(x)) = \sin(x)$

⑤ $y^{(5)} = \frac{d}{dx}[\sin(x)] = \cos(x)$

Note if $y = \cos x$, a similar pattern occurs

<u>Ex 3:</u> Given $f(x) = xe^x$. Find $f''(x)$ and $f^{(3)}(x)$.

① Find f'.

Product Rule $\quad u = x \quad v = e^x$

$\quad u' = 1 \quad v' = e^x$

$$f'(x) = u'v + uv'$$
$$= 1 \cdot e^x + xe^x$$
$$= (1 + x)e^x$$

② Find f''.

Product Rule: $\quad u = 1 + x \quad v = e^x$

$\quad u' = 1 \quad v' = e^x$

$$f''(x) = u'v + uv'$$
$$= 1 \cdot e^x + (1 + x)e^x$$
$$= (1 + 1 + x)e^x$$
$$= (2 + x)e^x$$

③ Find f'''.

Product Rule: $u = 2+x$ $\quad v = e^x$

$u' = 1$ $\quad v' = e^x$

$$f'''(x) = u'v + uv'$$
$$= 1e^x + (2+x)e^x$$
$$= (1+2+x)e^x$$
$$= (3+x)e^x$$

Ex 4: Find $f''(x)$ of $f(x) = \frac{4}{(x^2+1)^2}$

Rewrite $f(x) = 4(x^2+1)^{-2}$

① Find f'.

By Chain Rule,

$$f'(x) = 4(-2)(x^2+1)^{-3}(2x)$$
$$= \frac{-16x}{(x^2+1)^3}$$

② Find f''.

By Quotient Rule,

$u = -16x$ $\quad v = (x^2+1)^3$

$u' = -16$ $\quad v' = 3(x^2+1)^2(2x)$

$$f''(x) = \frac{u'v - uv'}{v^2} = \frac{-16(x^2+1)^3 - (-16x)(3)(x^2+1)^2(2x)}{((x^2+1)^3)^2}$$

$$= \frac{(-16)\cancel{(x^2+1)^2}\left[x^2+1 - x(3)(2x)\right]}{(x^2+1)^{\cancel{6}\,4}}$$

$$= \frac{-16(x^2+1-6x^2)}{(x^2+1)^4}$$

$$= \frac{-16(x^{\wedge}+1-6x^{\wedge})}{(x^2+1)^4}$$

$$= \frac{-16(-5x^2+1)}{(x^2+1)^4}$$

$s(t)$ $v(t)$

Position & Velocity & Acceleration Function

Recall that $v(t) = s'(t)$

Acceleration Function $[a(t)]$ tells us how fast the velocity changes

Hence $a(t) = v'(t) = [s'(t)]' = s''(t)$

Ex 5: The position function of a particle is

$$s(t) = \frac{1}{12}t^4 - \frac{4}{3}t^3 + 8t^2 - 64t$$

What is the acceleration of the particle @ $t=2$?

$$a(t) = s''(t) \quad @\ t=2$$

$$s'(t) = \frac{4}{12}t^3 - \frac{4}{3}\cdot 3t^2 + 8\cdot 2t - 64$$

$$= \frac{1}{3}t^3 - 4t^2 + 16t - 64$$

$$s''(t) = \frac{3}{3}t^2 - 4(2)t + 16 + 0$$

$$= t^2 - 8t + 16$$

$$a(2) = s''(2) = 2^2 - 8(2) + 16$$

$$= 4$$

Lesson 13: Implicit Differentiation

Explicit Form: $y = f(x)$

Implicit Form: When a function is NOT written in explicit form

ex. ① $y - 2x = 1$ ② $x^2 + y^2 = 2$ ③ $y^2 + y - 1 = x$

To differentiate functions of this kind, we use a technique called implicit differentiation.

Use this technique when solving for y is MESSY.

Ex 1: Use implicit differentiation to find slope of tangent line of

$$x^2 - y^2 = 4x + 8y \quad \text{at} \quad (0,0)$$

i.e. $\frac{dy}{dx}(0,0)$

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4x + 8y)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(8y)$$

$$2x \cancel{\frac{dx}{dx}}^{\,1} - 2y\frac{dy}{dx} = 4\cancel{\frac{dx}{dx}}^{\,1} + 8\frac{dy}{dx}$$

$$2x - 2y\frac{dy}{dx} = 4 + 8\frac{dy}{dx}$$

Solve for dy/dx.

$$2x - 4 = 8\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2x - 4 = (8 + 2y)\frac{dy}{dx}$$

$dy \quad 2x-4 \qquad 2(0)-4 \quad -4$

$$\frac{dy}{dx} = \frac{2x-4}{8+2y} \quad \Rightarrow \quad \frac{dy}{dx}(0,0) = \frac{2(0)-4}{8+2(0)} = \frac{-4}{8} = -\frac{1}{2}$$

Ex 2: Use implicit differentiation to find dy/dx of

$$yx^2 + e^y = x$$

$$\frac{d}{dx}(yx^2 + e^y) = \frac{d}{dx}(x)$$

$$\underbrace{\frac{d}{dx}(yx^2)} + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

Is this a function of x or y? Both

Is it a product or quotient? Product $\Rightarrow$ Product Rule

$$\frac{d}{dx}(y) \cdot x^2 + y \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$1 \cdot \frac{dy}{dx} \cdot x^2 + y \cdot 2x \cancel{\frac{dx}{dx}}^{1} + e^y \frac{dy}{dx} = 1 \cdot \cancel{\frac{dx}{dx}}^{1}$$

$$x^2 \frac{dy}{dx} + 2xy + e^y \frac{dy}{dx} = 1$$

Solve dy/dx

$$x^2 \frac{dy}{dx} + e^y \frac{dy}{dx} = 1 - 2xy$$

$$(x^2 + e^y) \frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{1-2xy}{x^2+e^y}$$

Extra Credit 2

Determine

ⓐ $\frac{d}{dx}(xy) = \frac{d}{dx}(x) \cdot y + x \cdot \frac{d}{dx}(y)$

ⓑ $\frac{d}{dx}\left(\frac{x}{y}\right)$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y) \qquad ② \ \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= 1\cdot \cancel{\frac{dx}{dx}}^{\,1}\cdot y + x\cdot 1\cdot\frac{dy}{dx}$$

Hint: Quotient Rule

$$\underbrace{= y + x\frac{dy}{dx}}_{\text{Basically product rule}}$$

Ex 3: Use implicit differentiation to find dy/dx of

$$4\sin(x)\cos(y) = 3$$

$$\frac{d}{dx}\left(4\sin(x)\cos(y)\right) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}\left(4\sin(x)\right)\cos(y) + 4\sin(x)\cdot\frac{d}{dx}\left(\cos(y)\right) = \frac{d}{dx}(3)$$

$$4\cos(x)\cancel{\frac{dx}{dx}}^{\,1}\cos(y) + 4\sin(x)\left[-\sin(y)\right]\frac{dy}{dx} = 0$$

$$4\cos(x)\cos(y) = 4\sin(x)\sin(y)\frac{dy}{dx}$$

$$\frac{4\cos(x)\cos(y)}{4\sin(x)\sin(y)} = \frac{dy}{dx}$$

$$\cot(x)\cot(y) = \frac{dy}{dx}$$

Ex 4: Use implicit differentiation to find dy/dx of

$$6\tan(2x+3y) = 11x$$

$$\underbrace{\frac{d}{dx}\left(6\tan(2x+3y)\right)}_{\text{Chain Rule}} = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\left[2\cancel{\frac{dx}{dx}}^{1} + 3\frac{dy}{dx}\right] = 11\cancel{\frac{dx}{dx}}^{1}$$

$$6\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = 11$$

$$2 + 3\frac{dy}{dx} = \frac{11}{6\sec^2(2x+3y)}$$

$$2 + 3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y)$$

$$3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y) - 2$$

$$\frac{dy}{dx} = \frac{11}{18}\cos^2(2x+3y) - \frac{2}{3}$$

Formal Proof of why $\frac{d}{dx}[\ln x] = \frac{1}{x}$

Let $y = \ln x$. Note $y = \ln x \iff e^y = x$

Differentiate

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx} = 1 \cdot \frac{dx}{dx}$$

$$e^y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Hence $\frac{d}{dx}[\ln x] = \frac{1}{x}$

SP26_MA16010_L14+15_Handout

Wednesday, February 18, 2026 1:31 PM

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MA 16010 LESSONS 14-15: RELATED RATES

This space is left for you to take your own notes.

Related rates are word problems that use implicit differentiation.

We will be taking the derivative of equations with respect to time, t.

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2 + b^2 = c^2$	Triangle $A = \frac{1}{2}bh$ $P = a + b + c$	Trapezoid $A = \frac{1}{2}(a + b)h$
Rectangular Box $V = lwh$ $S = 2(hl + lw + hw)$	Rectangle $A = lw$ $P = 2l + 2w$	Circle $A = \pi r^2$ $C = 2\pi r$
Right Circular Cylinder $V = \pi r^2 h$ $SA = 2\pi rh$	Sphere $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$	Cone $V = \frac{1}{3}\pi r^2 h$ $SA = \pi rl + \pi r^2$

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

(a) Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = -2$ and $y = 1$

$$\frac{d}{dt}(x+y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \underbrace{\frac{dx}{dt}}_{= -2} + 3\underbrace{y}_{y=1}^2 \cdot \frac{dy}{dt} = 0$$

$$-2 + 3(1)^2 \cdot \frac{dy}{dt} = 0$$

$$-2 + 3\frac{dy}{dt} = 0$$

$$3\frac{dy}{dt} = 2$$

$$\frac{dy}{dt} = \frac{2}{3}$$

(b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$

to determine y remember $x+y^3 = 2$ and $x = 1$

$$1 + y^3 = 2$$

$$y^3 = 1$$

$$y = 1$$

$$\frac{d}{dt}(x+y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

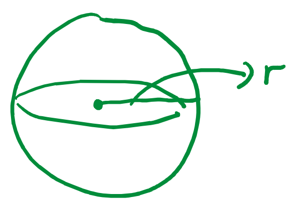
$$1 \cdot \underbrace{\frac{dx}{dt}}_{\text{WANT}} + 3y^2 \underbrace{\frac{dy}{dt}}_{= 3} = 0$$

$$\frac{dx}{dt} + 3(1)^2(3) = 0$$

$$\frac{dx}{dt} + 9 = 0 \Rightarrow \frac{dx}{dt} = -9$$

Example 2: A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20 \frac{cm^3}{s}$

WANT: $\left.\frac{dr}{dt}\right|_{r=12\,cm}$

Step 3: Find an equation relating the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$1 \cdot \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

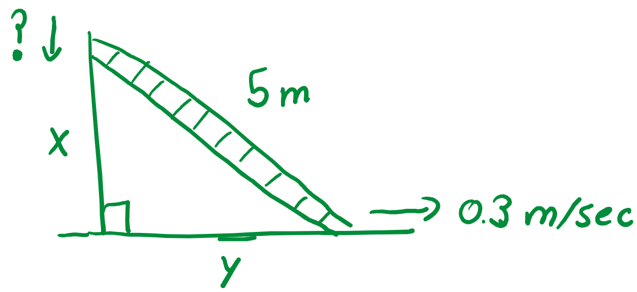
$$-20 = 4\pi(12)^2 \frac{dr}{dt}$$

$$\frac{-\cancel{20}^{\,5}}{\cancel{4}\pi(12)^2} = \frac{dr}{dt}$$

$$\frac{-5}{144\pi} = \frac{dr}{dt}$$

Example 3: A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3 m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



x is how tall from the wall

y is far from the tall

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 0.3\ \frac{m}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{y=3\ m}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + y^2 = 5^2 \iff x^2 + y^2 = 25$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(25)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

I need x. WANT 3 0.3

Find x by plugging y=3 into

$$x^2 + y^2 = 25$$
$$x^2 + 3^2 = 25$$
$$x^2 + 9 = 25$$
$$x^2 = 16$$
$$x = 4$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$2(4)\frac{dx}{dt} + 2(3)(0.3) = 0$$

$$2(4)\frac{dx}{dt} = -2(3)(0.3)$$

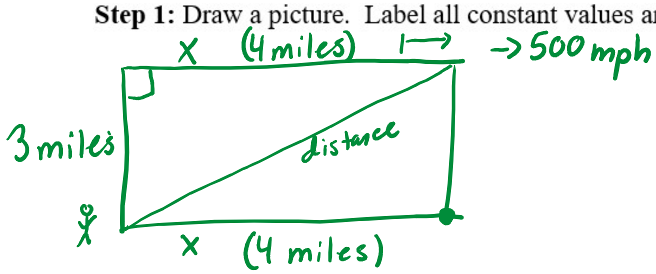
$$\frac{dx}{dt} = \frac{-\cancel{2}(3)(0.3)}{\cancel{2}(4)}$$

$$= \frac{-0.9}{4} = -0.225$$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(1) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



z - distance

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500$ mph

WANT: $\left.\frac{dz}{dt}\right|_{x=4 \text{ miles}}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + 3^2 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2 + 3^2) = \frac{d}{dt}(z^2)$$

$$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

$$\underset{4}{x}\underset{500}{\frac{dx}{dt}} = \overset{5}{z}\frac{dz}{dt}$$

Find z with $x^2 + 3^2 = z^2$ and $x = 4$

$$4^2 + 3^2 = z^2$$

$$16 + 9 = z^2$$

$$25 = z^2$$

$$z = 5$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$4(500) = 5\frac{dz}{dt}$$

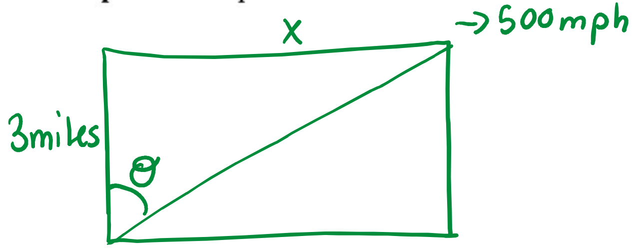
$$\frac{4(\overset{100}{\cancel{500}})}{\cancel{5}} = \frac{dz}{dt}$$

$$\frac{dz}{dt} = 400$$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(2) How fast is the angle of elevation changing when it is $\pi/3$?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500 \text{ mph}$ **WANT:** $\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3}$

Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{x}{3}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{3}\right)$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt} \to 500$$

$\pi/3$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\left(\sec\left(\frac{\pi}{3}\right)\right)^2 \frac{d\theta}{dt} = \frac{1}{3}\cdot 500$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

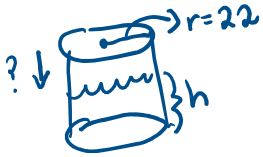
$$\sec\left(\frac{\pi}{3}\right) = 2$$

$$2^2 \frac{d\theta}{dt} = \frac{500}{3}$$

$$\frac{d\theta}{dt} = \frac{500}{3\cdot 4} = \frac{250}{3\cdot 2} = \frac{125}{3}$$

HW 15.5: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/*sec*? Note: The formula right circular cylinder is $V = \pi r^2 h$.

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -28 \frac{cm^3}{s}$ **WANT:** $\frac{dh}{dt}$

Step 3: Find an equation relating the relevant variables.

$$V = \pi r^2 h \iff V = \pi (22)^2 h$$

b/c radius doesn't change

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi \cdot 22^2 h)$$

$$1 \cdot \frac{dV}{dt} = \pi \cdot 22^2 \frac{dh}{dt}$$

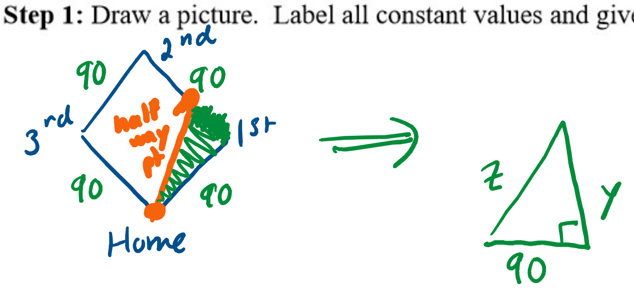
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$-28 = \pi \cdot 22^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-28}{\pi \cdot 22^2} = \frac{-4 \cdot 7}{\pi \cdot \cancel{2^2} \cdot 11^2} = \frac{-7}{121\pi}$$

HW 16.3: A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 14 ft/sec. At what rate is the player's distance from home base increasing when he is halfway from first to second base?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



y - Home to 1st

z - distance from home to half way pt

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 14 \ \frac{ft}{s}$

WANT: $\left.\frac{dz}{dt}\right|_{y = \frac{90}{2} = 45}$

Step 3: Find an equation relating the relevant variables.

$$90^2 + y^2 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(90^2 + y^2) = \frac{d}{dt}(z^2)$$

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$y \to 45$, $\frac{dy}{dt} \to 14$

To find z remember

$90^2 + y^2 = z^2$ & $y = 45$

$90^2 + 45^2 = z^2$

$45\sqrt{5} = z$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\cancel{2}(\cancel{45})(14) = \cancel{2}(\cancel{45})\sqrt{5}\,\frac{dz}{dt}$$

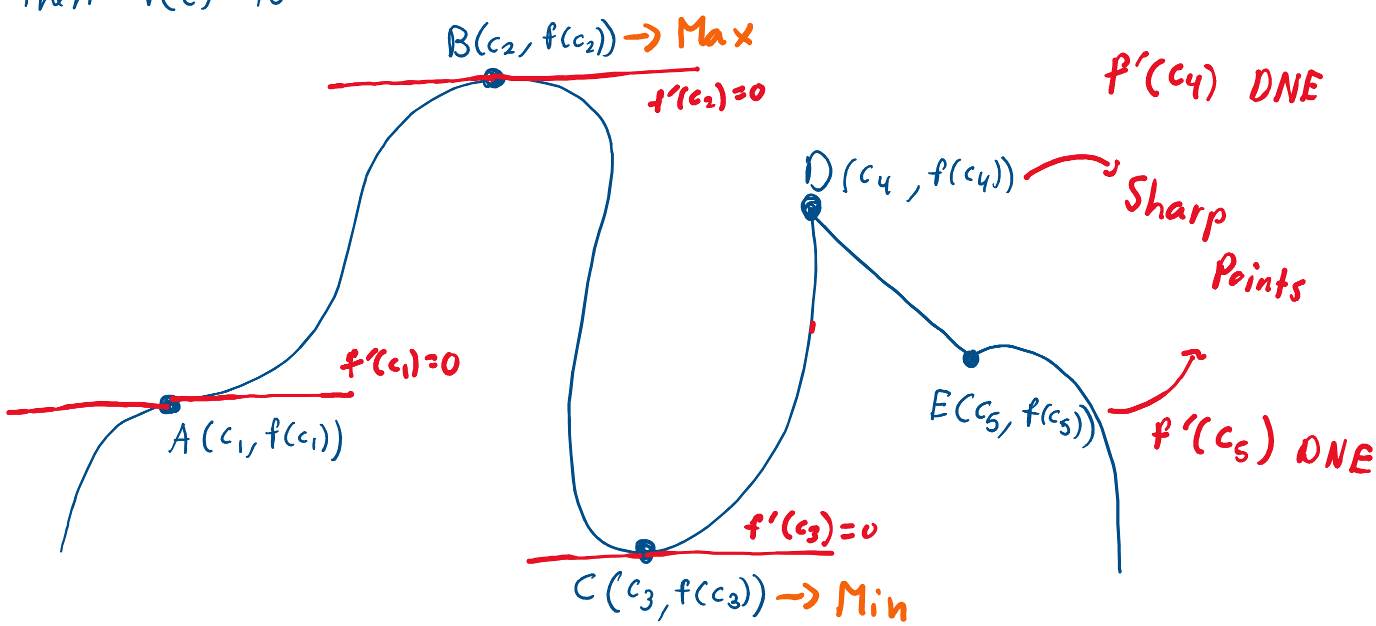
$$\frac{14}{\sqrt{5}} = \frac{dz}{dt}$$

Monday, February 23, 2026 1:29 PM

Lesson 16: Relative Extrema & Critical Numbers

<u>Def</u>: (a) If $f(c) \geq f(x)$ for all x in an open interval I containing c, then $f(c)$ is a relative max.

(b) If $f(c) \leq f(x)$ for all x in an open interval I containing c, then $f(c)$ is a relative minimum.



Overall, all relative extremas occur at points where derivative is zero or DNE.

Question: Is the reverse true?
i.e. If the derivative is zero or DNE, do we have a relative extrema?

No. Refer back to Points A and E from the graph.

<u>Def</u>: Let c be a # in the domain of f. If $f'(x) = 0$ or $f'(x)$ DNE at $x = c$, then c is a critical points.

<u>Ex 1</u>: Find the critical numbers of $y = x^4 - 2x^3$

i.e. Solve $y'(x) = 0$ for x.

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$$y' = 4x^3 - 2\cdot 3x^2 = 4x^3 - 6x^2 = 0$$

$$2x^2(2x-3) = 0$$

$$x = 0, \ 3/2$$

Since the domain of polynomial is $(-\infty, \infty)$ then $x = 0, 3/2$ are critical pt.

Ex 2: Find the critical numbers of $y = 5x^{4/5}$

i.e. Solve $y'(x) = 0$ for x.

$$y' = 5\cdot \frac{4}{5} x^{-1/5} = \frac{4}{\sqrt[5]{x}} = 0$$

$$\frac{4}{5\sqrt{x}} \cdot 5\sqrt{x} = 0 \cdot 5\sqrt{x}$$

$$4 = 0 \Rightarrow y' \neq 0 \text{ for all } x.$$

$y' = \frac{4}{\sqrt[5]{x}}$ DNE when $x = 0$.

Ex 3: Find the critical numbers of $y = 3x^4 e^x$.

Solve $y'(x) = 0$ for x.

y' is found by product rule.

$u = 3x^4 \qquad v = e^x$

$u' = 12x^3 \qquad v' = e^x$

$$y' = u'v + uv'$$
$$= 12x^3 e^x + 3x^4 e^x = 0$$

$$\underbrace{3x^3}_{x=0} \underbrace{e^x}_{\text{Never}} \underbrace{[4 + x]}_{x=-4} = 0$$

Domain is $(-\infty, \infty)$

So critical pt are $x = 0, -4$

$\underbrace{}_{x=0}$ $\quad$ Never $=0$. $\quad$ $\underbrace{}_{x=-4}$

Ex 4: Find the critical numbers of $f(x) = \underbrace{\ln(x^2+2x+1)}_{=\ln((x+1)^2)}$.

Solve $y'(x) = 0$ for x.

$$y' = \frac{1}{x^2+2x+1} \cdot \frac{d}{dx}(x^2+2x+1)$$

$$= \frac{1}{x^2+2x+1} \cdot (2x+2)$$

$$= \frac{2x+2}{x^2+2x+1}$$

$$= \frac{2\cancel{(x+1)}}{(x+1)^{\cancel{2}}} = \frac{2}{x+1} = 0$$

NEVER $f'(x)=0$.

Okay so when does $f'(x)$ DNE.

$$y' = \frac{2}{x+1}$$ DNE when $x=-1$.

$\Downarrow$

But not in domain of $f(x)$.

So no critical points.

Ex 5: Is $\pi/8$ a critical number for $y = 2\sin(2x) - 2\sqrt{2}x$?

Check if $y'(\pi/8) \stackrel{?}{=} 0$

$$y' = 2\cos(2x) \cdot \frac{d}{dx}(2x) - 2\sqrt{2}$$

$$= 2\cos(2x) \cdot 2 - 2\sqrt{2}$$

Plug $x = \frac{\pi}{8}$

$$y'\left(\frac{\pi}{8}\right) = 2\cos\left(\frac{2\pi}{8}\right) \cdot 2 - 2\sqrt{2}$$

$= 4\cos\left(\frac{\pi}{4}\right) - 2\sqrt{2}$

$= 4\frac{\sqrt{2}}{2} - 2\sqrt{2}$

$= 2\sqrt{2} - 2\sqrt{2} = 0$ ✓

Ex 6: Is $x = \frac{\pi}{12}$ a critical point of $y = 3\tan(3x) - \cos(3x)$?

Find y' and then plug $x = \frac{\pi}{12}$. Check to see if $= 0$.

$$y' = 3\sec^2(3x) \cdot 3 - (-\sin(3x)) \cdot 3$$
$$= 9\sec^2(3x) + 3\sin(3x)$$

$$y'\left(\frac{\pi}{12}\right) = 9\sec^2\left(\frac{3\pi}{12}\right) + 3\sin\left(\frac{3\pi}{12}\right)$$
$$= 9\sec^2\left(\frac{\pi}{4}\right) + 3\sin\left(\frac{\pi}{4}\right)$$
$$= 9 \cdot 2 + 3\frac{\sqrt{2}}{2}$$
$$\neq 0$$

$x = \pi/12$ is no a critical point.

$$\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}}$$
$$\sec^2\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2$$