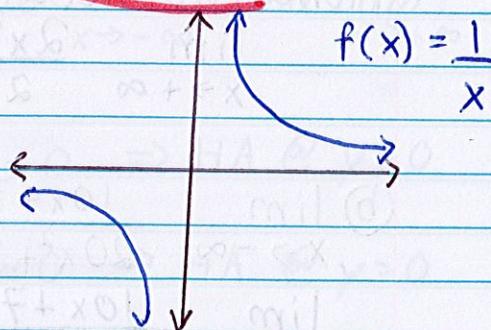# Lesson 21: Limits at Infinity

Recall

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$f(x) = \frac{1}{x}$$

Now we want

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \qquad \lim_{x \to +\infty} \frac{1}{x} = 0$$

The purpose of revisiting this topic is to determine
① End Behavior
② If we have a Horizontal/Slant Asymptote.

Example 1: Find the following limits

ⓐ $\lim\limits_{x \to +\infty} \dfrac{3}{x} = 3 \lim\limits_{x \to +\infty} \dfrac{1}{x} = 3 \cdot 0 = 0$

ⓑ $\lim\limits_{x \to -\infty} \left(\dfrac{x}{3} + 2 \right) = \dfrac{-\infty}{3} + 2 = -\infty + 2 = -\infty$

ⓒ $\lim\limits_{x \to +\infty} \left(\dfrac{x}{2} + \dfrac{5}{x} \right) = \dfrac{+\infty}{2} + 5 \cdot 0 = \dfrac{\infty}{2} = \infty$

General Rule: The limit of a rational function $f(x) = \dfrac{p(x)}{q(x)}$
as $x \to \pm\infty$ is determined by the leading terms of the numerator and the denominator.

Recall: A leading term of a polynomial is the term that has the highest power of x.

Example 2: Find the following limits:

ⓐ $\lim\limits_{x \to +\infty} \dfrac{2x^2 + 1}{x^2 - 1}$

$\lim\limits_{x \to +\infty} \dfrac{2x^2 + 1}{x^2 - 1} = \dfrac{"\infty"}{\infty} \Rightarrow$ No-no.

Hence we need to try using the General Rule,

$$\lim_{x \to +\infty} \frac{2x^2}{2} = \lim_{x \to +\infty} 2 = 2$$

(b) $\displaystyle\lim_{x \to -\infty} \frac{10x+7}{20x^2+3}$

$$\lim_{x \to -\infty} \frac{10x+7}{20x^2+3} = \frac{-\infty}{\infty} \Rightarrow \text{No-no}$$

Hence we need to try using the General Rule,

$$\lim_{x \to -\infty} \frac{10x}{20x^2} = \lim_{x \to -\infty} \frac{1}{2x} = \frac{1}{2} \cdot \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{2} \cdot 0 = 0$$

(c) $\displaystyle\lim_{x \to +\infty} \frac{5x^3+9}{2x^2+1}$

$$\lim_{x \to +\infty} \frac{5x^3+9}{2x^2+1} = \frac{\infty}{\infty} \Rightarrow \text{No-no}$$

Hence we need to try using the General Rule,

$$\lim_{x \to +\infty} \frac{5x^3}{2x^2} = \lim_{x \to +\infty} \frac{5}{2} \cdot x = \frac{5}{2} \cdot \infty = \infty$$

Horizontal Asymptotes

The line $y=L$, where L is a constant, is a horizontal asymptote of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to +\infty} f(x) = L$$

Note both limits need not match.
For example when $f(x) = e^x$

Example 3: Find, if they exist, the horizontal asymptote of the following functions:

(a) $h(x) = \dfrac{x-4}{x^2-1}$

To determine the HA we need to find $\lim\limits_{x \to -\infty}$ and $\lim\limits_{x \to +\infty}$

with the general rule.

$$\lim\limits_{x \to -\infty} h(x) = \lim\limits_{x \to -\infty} \frac{x}{x^2} = \lim\limits_{x \to -\infty} \frac{1}{x} = 0 \quad \Rightarrow \text{HA @ } y = 0$$

$$\lim\limits_{x \to +\infty} h(x) = \lim\limits_{x \to +\infty} \frac{x}{x^2} = \lim\limits_{x \to +\infty} \frac{1}{x} = 0 \quad \Rightarrow \text{HA @ } y = 0$$

ⓑ $h(x) = \dfrac{x^3 + 5}{2x + 1}$

To determine the HA we need to find $\lim\limits_{x \to -\infty}$ and $\lim\limits_{x \to \infty}$

with the general rule.

$$\lim\limits_{x \to -\infty} h(x) = \lim\limits_{x \to -\infty} \frac{x^3}{2x} = \lim\limits_{x \to -\infty} \frac{x^2}{2} = \infty$$

$$\lim\limits_{x \to +\infty} h(x) = \lim\limits_{x \to +\infty} \frac{x^3}{2x} = \lim\limits_{x \to +\infty} \frac{x^2}{2} = \infty$$

Since each limit don't equal a constant, there are NO HA!

Slant Asymptotes

The line $y = ax + b$, where a and b are constants and $a \neq 0$, is a slant asymptote of $f(x)$ if $f(x)$ gets closer and closer to $y = ax + b$ as $x \to \pm\infty$

To find the slant asymptotes of a rational function, we use
  • Synthetic Division   • Long Division