

Lesson 22: A Summary of Curve Sketching

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Now we are going to assemble what we learned to analyze and sketch rational functions.

Ex 1: Analyze and sketch the graph of $f(x) = \frac{x^2 - x}{x - 3}$

① Domain of f: When does $f(x)$ DNE?

Fractions are undefined when denominator = 0

$$x - 3 = 0$$

$$x = 3 \Rightarrow \text{Domain: } (-\infty, 3) \cup (3, \infty)$$

② x-intercept: Set $y = 0$. Solve for x .

$$\frac{0}{1} = \frac{x^2 - x}{x - 3}$$

$$0(x - 3) = x^2 - x$$

$$0 = x^2 - x$$

$$0 = x(x - 1)$$

$$\begin{array}{l|l} 0 = x & 0 = x - 1 \\ & x = 1 \end{array}$$

x-intercepts: $(0, 0), (1, 0)$

③ y-intercept: Set $x = 0$. Solve for y .

$$y = \frac{0^2 - 0}{0 - 3} = \frac{0}{-3} = 0$$

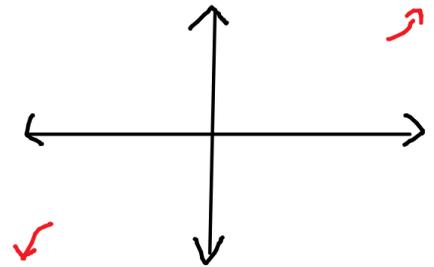
y-intercept: $(0, 0)$

$$f(x) = \frac{x^2 - x}{x - 3}$$

④ End Behavior: Use the General Rule

$$\textcircled{a} \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2}{x} = \lim_{x \rightarrow -\infty} x = -\infty$$

$$\textcircled{b} \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x^2}{x} = \lim_{x \rightarrow +\infty} x = +\infty$$



$$f(x) = \frac{x^2 - x}{x - 3} = \frac{x(x-1)}{x-3}$$

⑤ Asymptotes

① Vertical Asymptote:

① check that $f(x)$ is simplify. ✓

② Set denominator to 0. Solve.

$$x - 3 = 0$$

$$x = 3 \Rightarrow \underline{\text{VA}}: x = 3$$

② Horizontal Asymptote: Use ④

Since $\lim_{x \rightarrow \pm\infty} f(x) \neq L$, where L is a constant, then there is no HA.

③ Slant Asymptote

① check that difference b/w the powers of the leading terms of numerator and denominator is equal to 1. ✓

② If so, use Synthetic Division or Long Division.

$$\begin{array}{r|rrr} 3 & 1 & -1 & 0 \\ & \downarrow & 3 & 6 \\ \hline & 1 & 2 & 6 \end{array}$$

$$\Rightarrow f(x) = \boxed{x+2} + \frac{6}{x-3}$$

↓
Slant Asymptote
@ $y = x + 2$

⑥ Critical #s: $f'(x) = 0$ and $f'(x)$ DNE

$$\begin{array}{ll} u(x) = x^2 - x & v(x) = x - 3 \\ u'(x) = 2x - 1 & v'(x) = 1 \end{array}$$

$$\begin{aligned} f'(x) &= \frac{(2x-1)(x-3) - (x^2-x)}{(x-3)^2} \\ &= \frac{2x^2 - 7x + 3 - x^2 + x}{(x-3)^2} \end{aligned}$$

	$2x$	-1
x	$2x^2$	$-x$
-3	$-6x$	3

$$= \frac{x^2 - 6x + 3}{(x-3)^2} = 0$$

$x^2 - 6x + 3 = 0 \leftarrow$ Use Quadratic Formula

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 12}}{2} \\ &= \frac{6 \pm \sqrt{24}}{2} = \frac{6 \pm 2\sqrt{6}}{2} = 3 \pm \sqrt{6} \end{aligned}$$

$$f'(x) = \frac{x^2 - 6x + 3}{(x-3)^2} \text{ DNE when } (x-3)^2 = 0 \\ x = 3$$

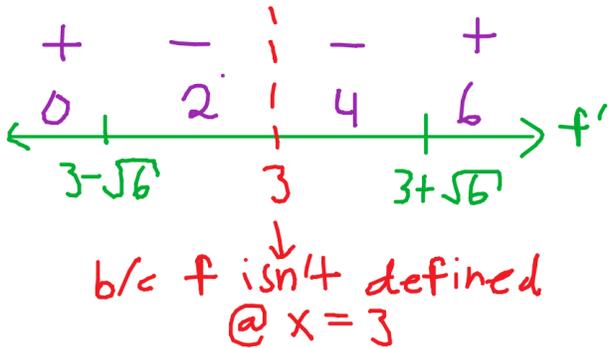
Recall domain of f is $(-\infty, 3) \cup (3, \infty)$. So $x=3$ can't be a critical #.

Critical #s: $x = 3 \pm \sqrt{6}$

$$f(x) = \frac{x^2 - x}{x - 3} \quad f'(x) = \frac{x^2 - 6x + 3}{(x - 3)^2}$$

⑦ Increasing/Decreasing

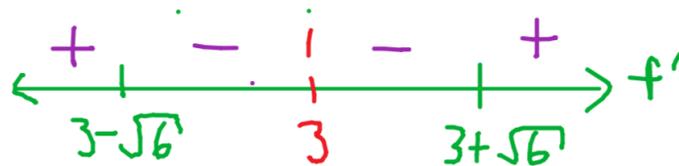
Note denominator is always positive.



Increasing:
 $(-\infty, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, \infty)$

Decreasing:
 $(3 - \sqrt{6}, 3) \cup (3, 3 + \sqrt{6})$
 $= (3 - \sqrt{6}, 3 + \sqrt{6})$

⑧ Relative Extrema: Use ⑦



By First Derivative Test,

Rel max: $x = 3 - \sqrt{6}$

Rel min: $x = 3 + \sqrt{6}$

$$f(x) = \frac{x^2 - x}{x - 3} \quad f'(x) = \frac{x^2 - 6x + 3}{(x - 3)^2}$$

⑨ Concavity: $f''(x) = 0$ and $f''(x)$ DNE

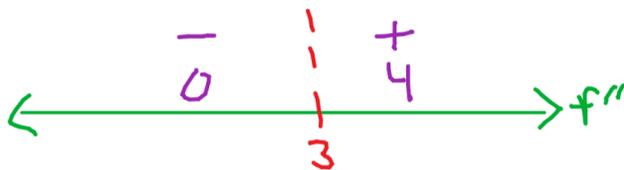
$$\begin{aligned} u(x) &= x^2 - 6x + 3 & v(x) &= (x - 3)^2 \\ u'(x) &= 2x - 6 & v'(x) &= 2(x - 3) \\ &= 2(x - 3) & & \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{2(x - 3)(x - 3)^2 - 2(x - 3)(x^2 - 6x + 3)}{(x - 3)^4} \\ &= \frac{2(x - 3) [(x - 3)^2 - (x^2 - 6x + 3)]}{(x - 3)^4} \\ &= \frac{2 [x^2 - 6x + 9 - x^2 + 6x - 3]}{(x - 3)^3} = \frac{2 \cdot 6}{(x - 3)^3} = \frac{12}{(x - 3)^3} = 0 \end{aligned}$$

$$f''(x) = \frac{12}{(x - 3)^3} = 0 \Rightarrow f''(x) \neq 0 \text{ b/c numerator } \neq 0$$

$$f''(x) \text{ DNE when } (x - 3)^3 = 0 \\ x = 3$$

But we know $f(x)$ DNE @ $x = 3$



Concave Up: $(3, \infty)$
Concave Down: $(-\infty, 3)$

⑩ Inflection Points: Use ⑨ and check for sign change



Is there a change? **Yes**

But $f(x)$ isn't defined @ $x = 3$, so no inflection points.

① Graph

$$f(x) = \frac{x^2 - x}{x - 3}$$

