

MA 16010 LESSONS 23-25: OPTIMIZATION

Optimization Problems: Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- e.g. The minimum cost for constructing some object,
- e.g. The maximum profit to gain for a business, and so on.

How do we solve an optimization problem?

- Determine a function (**known as objective function**) that we need to maximize or minimize.
 - Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
 - If there are constraint equations, rewrite the **objective function** as a function of only one variable.
 - Then we can solve for absolute maximum or minimum like we did before.
 - Using either the First Derivative Test or Second Derivative Test.
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Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Step 6: Reread the question and be sure you have answered exactly what was asked.

Example 1: Of all the numbers whose sum is 50, find the two that have the maximum product.

Let x and y be such #s.

Step 1: Identify what quantity you are trying to optimize. Product, P

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

N/A

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$P = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$x + y = 50$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y ,

$$y = 50 - x$$

Plug y into (3).

$$P = xy$$

$$= x(50 - x)$$

$$= 50x - x^2$$

Find P' and set $= 0$,

$$P' = 50 - 2x = 0$$

$$50 = 2x$$

$$25 = x$$

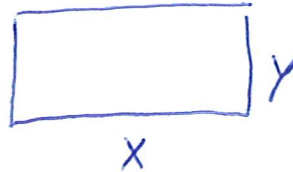
Step 6: Reread the question and be sure you have answered exactly what was asked.

$$\begin{aligned} x = 25 &\Rightarrow y = 50 - x \\ &= 50 - 25 \\ y &= 25 \end{aligned}$$

Example 2: A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$2x + 2y = 54 \quad \Leftrightarrow \quad x + y = 27$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y .

$$y = 27 - x$$

Plug y into (3)

$$\begin{aligned} A &= x(27 - x) \\ &= 27x - x^2 \end{aligned}$$

Find A' and set $= 0$.

$$\begin{aligned} A' &= 27 - 2x = 0 \\ x &= 27/2 \end{aligned}$$

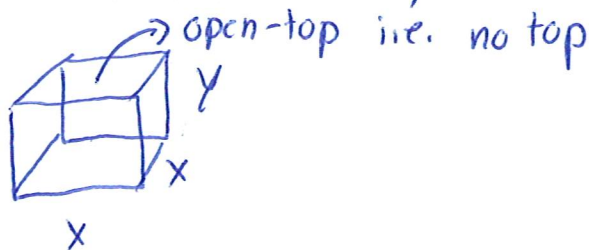
Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = \frac{27}{2} \Rightarrow \left. \begin{array}{l} y = 27 - x \\ y = \frac{27}{2} \end{array} \right\} \text{Area} = \frac{27}{2} \times \frac{27}{2} = \frac{749}{4}$$

Example 3: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

Step 1: Identify what quantity you are trying to optimize. Surface Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = x^2 + 4xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$8 = V = x^2 y$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y .

$$\frac{8}{x^2} = y$$

Plug y into (3).

$$A = x^2 + 4x \left(\frac{8}{x^2} \right) \\ = x^2 + \frac{32}{x}$$

Rewrite A to be
 $A = x^2 + 32x^{-1}$

Find A' and set $= 0$.
 $A' = 2x - 32x^{-2} = 0$

$$2x - \frac{32}{x^2} = 0$$

$$\frac{2x}{1} = \frac{32}{x^2}$$

$$2x^3 = 32$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

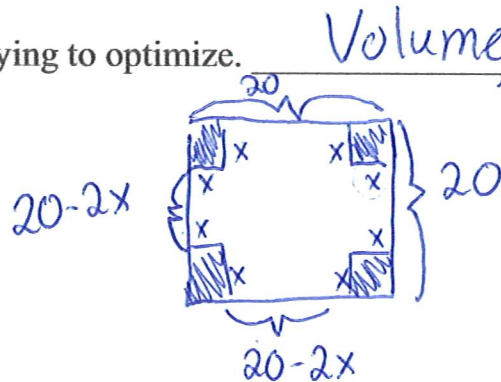
$$x = \sqrt[3]{16} \Rightarrow y = \frac{8}{x^2} \\ = \frac{8}{16^{2/3}}$$

Dimensions: $\sqrt[3]{16} \times \sqrt[3]{16} \times \frac{8}{16^{2/3}}$

Example 4: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x(20-2x)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

None since V is in terms of x only.

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Find V' and set $= 0$.

$$V' = 1 \cdot (20-2x)^2 + x \cdot 2(20-2x) \cdot (-2) = 0$$

$$(20-2x)[20-2x-4x] = 0$$

$$(20-2x)(20-6x) = 0$$

$$20-2x=0 \quad 20-6x=0$$

$$\cancel{x=10} \quad x = \frac{20}{6} = \frac{10}{3}$$

b/c if $x=10$, we will have no length or width.

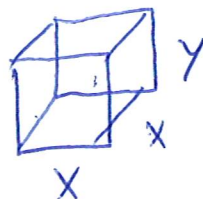
Step 6: Reread the question and be sure you have answered exactly what was asked.

$$\left. \begin{array}{l} x = \text{height} = \frac{10}{3} \\ \text{width} = \text{length} = 20 - 2x \\ \quad = 20 - 2\left(\frac{10}{3}\right) \\ \quad = \frac{40}{3} \end{array} \right\} \begin{array}{l} \text{Dimensions: } \frac{10}{3} \times \frac{40}{3} \times \frac{40}{3} \\ \text{Volume: } \frac{16000}{27} \end{array}$$

Example 5: A rectangular box has a square base. If the sum of the height and the perimeter of the square base is 20 in, what is the maximum possible volume?

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x^2 y$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$y + 4x = 20$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y .

$$y = 20 - 4x$$

Plug y into (3).

$$\begin{aligned} V &= x^2(20 - 4x) \\ &= 20x^2 - 4x^3 \end{aligned}$$

Find V' and set $= 0$.

$$V' = 40x - 12x^2 = 0$$

$$4x(10 - 3x) = 0$$

$$\begin{array}{ll} 4x = 0 & 10 - 3x = 0 \\ \cancel{x = 0} & x = \frac{10}{3} \end{array}$$

b/c if not we have no base.

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$\begin{aligned} x = \frac{10}{3} \Rightarrow y &= 20 - 4x \\ &= 20 - 4\left(\frac{10}{3}\right) \\ &= \frac{20}{3} \end{aligned}$$

$$\begin{aligned} \text{Volume} &= x^2 y \\ &= \left(\frac{10}{3}\right)^2 \cdot \frac{20}{3} = \frac{2000}{27} \end{aligned}$$

Example 6: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

b) What price, p , should the company charge to maximize their profit?

Step 1: Identify what quantity you are trying to optimize. Profit, P

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$P = R - C = pq - 5q = (p - 5)q$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$q = 2400 - 200p$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug (4) into (3).

$$\begin{aligned} P &= (p - 5)(2400 - 200p) \\ &= -200(p - 5)(p - 12) \\ &= -200(p^2 - 17p + 60) \end{aligned}$$

Find P' and set $= 0$.

$$\begin{aligned} P' &= -200(2p - 17) = 0 \\ p &= \frac{17}{2} = 8.5 \end{aligned}$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$p = \$8.50$$

Example 6: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

a) What price, p , should the company charge to maximize their revenue?

Step 1: Identify what quantity you are trying to optimize. Revenue, R

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$R = pq$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$q = 2400 - 200p$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug q into (3).

$$\begin{aligned} R &= p(2400 - 200p) \\ &= 2400p - 200p^2 \end{aligned}$$

Find R' and set $= 0$.

$$\begin{aligned} R' &= 2400 - 400p = 0 \\ p &= 6 \end{aligned}$$

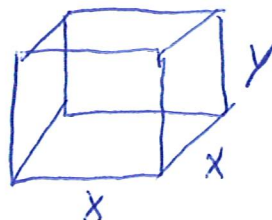
Step 6: Reread the question and be sure you have answered exactly what was asked.

\$6

Example 7: A rectangular box is to have a square base and a volume of 800 ft^3 . If the material for the base costs \$2 per square foot, the material for the sides costs \$4 per square foot, and the material for the top costs \$1 per square foot, determine the minimum cost for constructing such a box.

Step 1: Identify what quantity you are trying to optimize. Cost, C

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$C = \$2 \cdot x^2 + \$4(4xy) + \$1 \cdot x^2 = 3x^2 + 16xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$800 = x^2 y$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Solve (4) for y .

$$y = \frac{800}{x^2}$$

Plug y into (3).

$$C = 3x^2 + 16x \left(\frac{800}{x^2} \right)$$

$$= 3x^2 + \frac{12800}{x}$$

$$= 3x^2 + 12800x^{-1}$$

Find C' and set $= 0$.

$$C' = 6x - 12800x^{-2} = 0$$

$$6x - \frac{12800}{x^2} = 0$$

$$6x = \frac{12800}{x^2}$$

$$6x^3 = 12800$$

$$x^3 = \frac{6400}{3} \Rightarrow x = \sqrt[3]{\frac{6400}{3}}$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

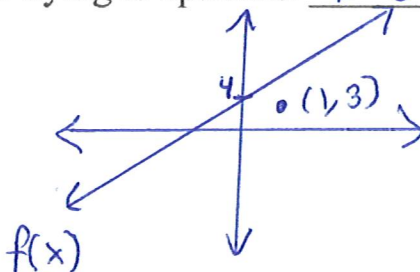
$$x = \sqrt[3]{\frac{6400}{3}} \Rightarrow y = \frac{800}{\left(\frac{6400}{3}\right)^{2/3}} \left. \vphantom{x} \right\} C = \$1491.47$$

Example 8: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.

Step 1: Identify what quantity you are trying to optimize.

Distance, D

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$D = (x-1)^2 + (y-3)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$y = 2x + 4$$

Step 5: Find the absolute extrema of the variable to be optimized on this domain.

Plug ④ into ③,

$$\begin{aligned} D &= (x-1)^2 + (2x+4-3)^2 \\ &= (x-1)^2 + (2x+1)^2 \end{aligned}$$

Find D' and set $= 0$,

$$\begin{aligned} D' &= 2(x-1) + 2(2x+1) \cdot 2 = 0 \\ 2x-2 + 4(2x+1) &= 0 \\ 2x-2 + 8x+4 &= 0 \\ 10x+2 &= 0 \\ x &= -\frac{1}{5} \end{aligned}$$

Step 6: Reread the question and be sure you have answered exactly what was asked.

$$x = -\frac{1}{5} \Rightarrow \left. \begin{aligned} y &= 2x + 4 \\ &= \frac{18}{5} \end{aligned} \right\} \left(-\frac{1}{5}, \frac{18}{5}\right)$$