Lesson 28: Area and Riemann Sums

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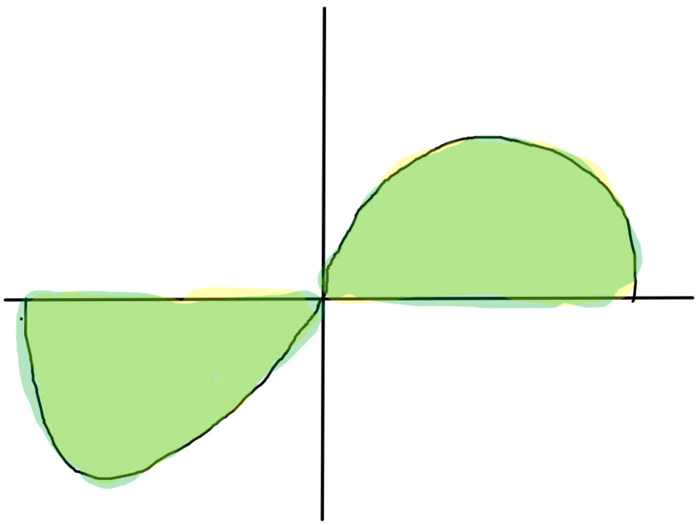


Purpose: How to approx the signed area under the curve of a function

Ex 1: Evaluate $\sum_{i=1}^{5} i^2$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Ex 2: Use the sigma notation to write the sum

$$2(1^3+1) + 2(2^3+1) + \ldots + 2(n^3+1)$$

$$= \sum_{i=1}^{n} 2(i^3+1)$$

Ex 3: Use the sigma notation to write the sum

$$\frac{2}{0+5} + \frac{2}{1+5} + \cdots + \frac{2}{n+5}$$

$$= \sum_{i=0}^{n} \frac{2}{i+5}$$

Signed Area is the area enclosed by the function and the x-axis with a sign.

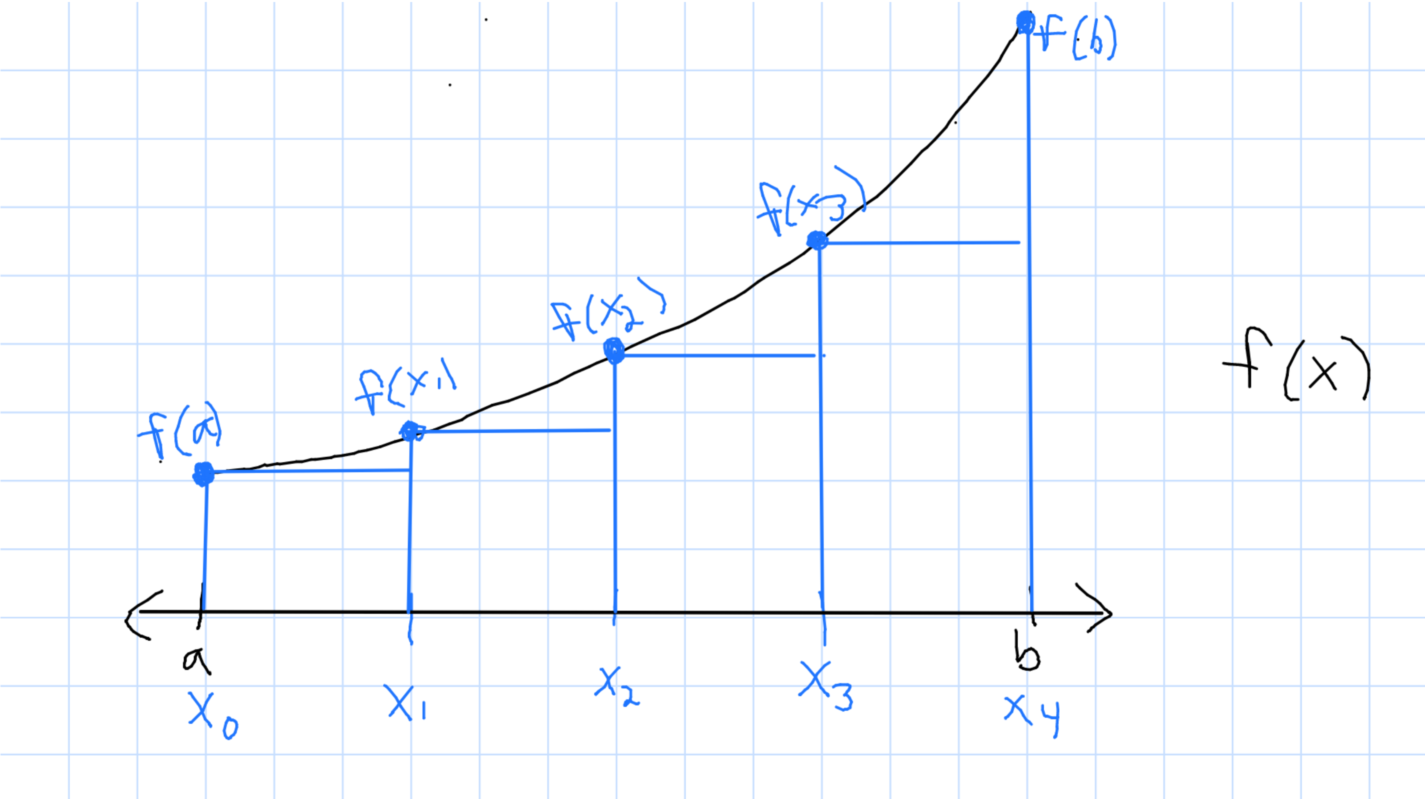
If the function is above the x-axis, then the area is postive.

If the function is below the x-axis, then the area is negative

We want to estimate the area under f(x) from a to b by using 4 rectangles.

We divide the interval [a,b] into 4 sub-intervals.

Then construct the rectangles by starting at the pts on f(x) whose x coordinates are the left end of each of the sub-intervals.



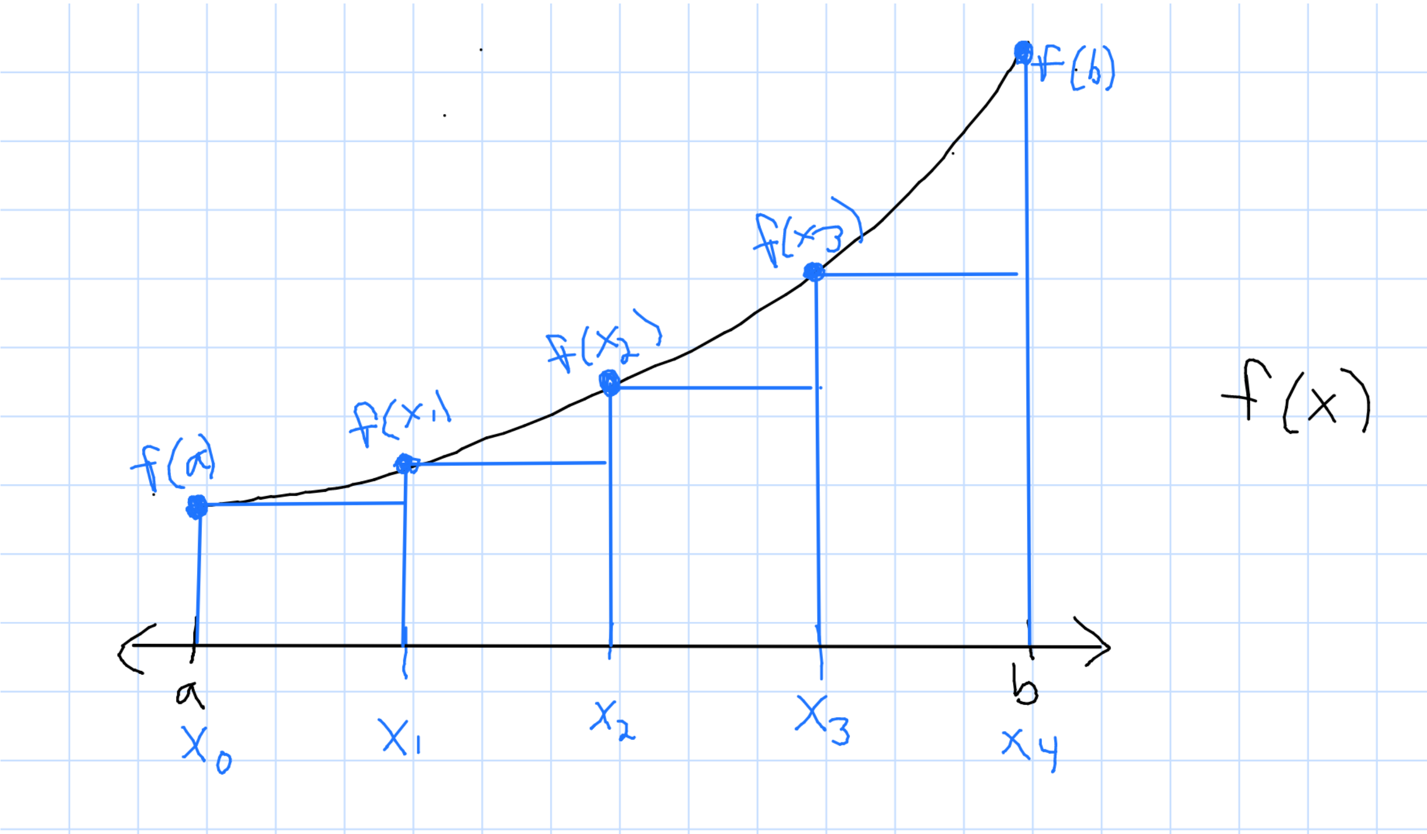
The total area of these rectangles gives us an estimate of the area under $f(x)$.

A sum like this is called the Left Riemann Sum.

How do we compute this sum?

Find the area of each rectangle and sum them all up.

But let's try to work smart



Width of each rectangle is the same.

$$w = \Delta x = \frac{b-a}{4}$$

Length of first rectangle $f(a) = f(x_0)$

second rectangle $f(x_1)$

$\vdots$

Length of ith rectangle $f(x_i)$

Area of each rectangle is

$$lw = \Delta x \, f(x_i)$$

4 rectangles $\Rightarrow \sum_{i=0}^{3} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$$\Delta x = \frac{b-a}{n}$$

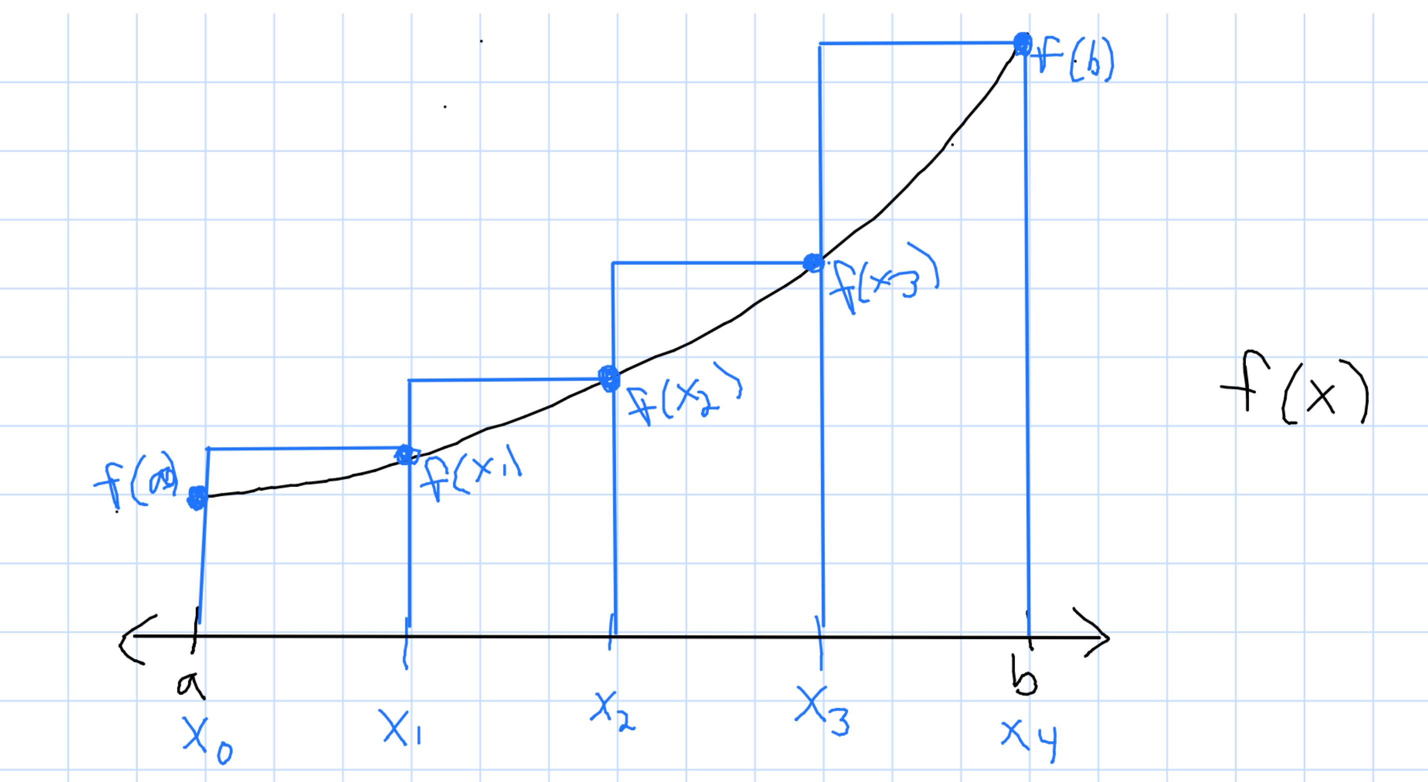
Why 3 not 4!?
Then construct the rectangles by starting at the pts on f(x) whose x coordinates are the left end of each of the sub-intervals.

Let's devise a formula for n rectangles.

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x \quad \text{where } x_i = a + i\Delta x$$
$$\Delta x = \frac{b-a}{n}$$

Can we do this by using the right end of each sub-interval?

Yep It's called Right Riemann Sum



$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } x_i = a + i\Delta x$$
$$\Delta x = \frac{b-a}{n}$$

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$$\Delta x = \frac{b-a}{n}$$

Note the only difference is the index.

Ex 4: Use the Left Riemann Sum with 3 rectangles to estimate the area under the curve $y=x^2$ on the interval of $[1,7]$.

$a = 1 \quad n = 3$

$b = 7$

$\Delta x = \frac{b-a}{n} = \frac{7-1}{3} = \frac{6}{3} = 2$

$x_i = a + i\Delta x = 1 + 2i$

$f(x_i) = (1+2i)^2 = 1 + 4i + 4i^2$

$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$\Delta x = 2$

$x_i = 1 + 2i$

$f(x_i) = 4i^2 + 4i + 1$

$n = 3$

$L_3 = \sum_{i=0}^{3-1} (4i^2+4i+1)\cdot 2$

$= 2 \sum_{i=0}^{2} (4i^2+4i+1)$

$= 2(4(0)^2+4(0)+1) + 2(4(1)^2+4(1)+1)$
$+ 2(4(2)^2+4(2)+1)$

$= 70$

Ex 5: Use the Right Riemann Sum with 4 rectangles to estimate the area under the curve $y=\sqrt{x}-x$ on the interval of $[0,4]$.

$a=0 \quad n=4$

$b=4$

$\Delta x=\frac{b-a}{n}=\frac{4-0}{4}=1$

$x_i=a+i\Delta x=0+i(1)=i$

$f(x_i)=\sqrt{i}-i$

$R_n=\sum_{i=1}^{n} f(x_i)\Delta x$

$\Delta x=1$

$x_i=i$

$f(x_i)=\sqrt{i}-i$

$n=4$

$R_4=\sum_{i=1}^{4}\left[\sqrt{i}-i\right]$

$=\cancel{(\sqrt{1}-1)}^{0}+(\sqrt{2}-\cancel{2})+(\sqrt{3}-3)+(\cancel{\sqrt{4}}-4)$

$=\sqrt{2}+\sqrt{3}-7$

Ex 6: Use the Right Riemann Sum with 50 rectangles to estimate the area under the curve $y = x^2 - 2x + 1$ on the interval of $[0, 10]$. Write your answer using sigma notation.

$a = 0$

$b = 10$ $\quad n = 50$

$\Delta x = \frac{b-a}{n} = \frac{10-0}{50} = \frac{10}{50} = \frac{1}{5}$

$x_i = a + i\Delta x = 0 + i \cdot 1/5 = i/5$

$f(x_i) = (i/5)^2 - 2(i/5) + 1$

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$\Delta x = 1/5$

$x_i = i/5$

$f(x_i) = \left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1$

$n = 50$

$$R_{50} = \sum_{i=0}^{50} \left[\left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1\right] \cdot \frac{1}{5}$$