Lesson 2: Finding Limits Numerically; One-Sided Limits

Lesson 2

Finding Limits Numerically

Definition: If $f(x)$ approaches ($\to$) L as $x \to c$ we say that the limit of $f(x)$ as $x \to c$ is L.

i.e. $\lim_{x \to c} f(x) = L$

Note that f does not need to be defined @ $x = c$ for the limit to exist.

Definition: If $f(x)$ inc or dec w/o bound as $x \to c$, then $\lim_{x \to c} f(x)$ is an infinite limit.

- If $f(x)$ inc w/o bound, $\lim_{x \to c} f(x) = +\infty$
- If $f(x)$ dec w/o bound, $\lim_{x \to c} f(x) = -\infty$

Finding Limits Numerically

$$\lim_{x \to c} f(x)$$

We evaluate f(x) at values of x that are getting closer and closer to c and see what happens with the values of the function.

Ex 1: Evaluate $\lim_{x \to 4} (2x-3)$ numerically

Recall $\lim_{x \to c} f(x)$. What is f(x)? $f(x) = 2x-3$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
f(x)	4.8	4.98	4.998	–	5.002	5.02	5.2

$\longrightarrow 5 \longleftarrow$

Hence $\lim_{x \to 4} (2x-3) = 5$

Ex 2: Evaluate $\lim_{x \to 3} \frac{x^3-3x^2}{x-3}$ numerically

Recall $\lim_{x \to c} f(x)$. What is f(x)? $f(x) = \frac{x^3-3x^2}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
f(x)	8.41	8.9401	8.994	–	9.006	9.0601	9.61

$\longrightarrow 9 \longleftarrow$

Hence $\lim\limits_{x \to 3} \dfrac{x^3 - 3x^2}{x - 3} = 9$

Ex 3: Given $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq -4 \\ 2 & \text{if } x = -4 \end{cases}$

Evaluate $\lim\limits_{x \to -4} f(x)$ numerically

What is $f(x)$ when $x \neq -4$? $f(x) = x^2 + 1$

x	-4.1	-4.01	-4.001	-4	-3.999	-3.99	-3.9
$f(x)$	17.81	17.0801	17.0080	—	16.992	16.9201	16.21

$\longrightarrow 17 \longleftarrow$

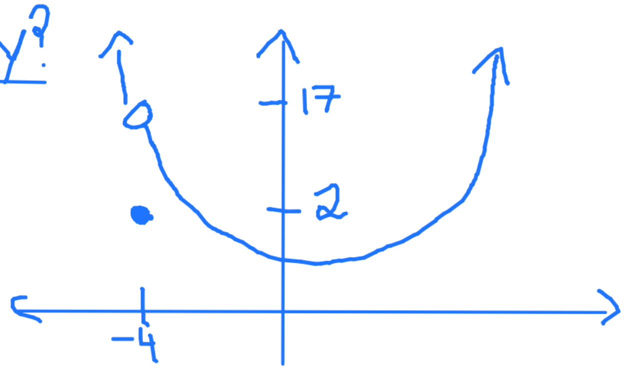
Hence $\lim\limits_{x \to -4} f(x) = 17$

Ex 3: Given $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq -4 \\ 2 & \text{if } x = -4 \end{cases}$

Questions: What is $f(-4)$? $\underline{\ 2\ }$

Is that equal to $\underbrace{\lim\limits_{x \to -4} f(x)}_{17}$? $\underline{\text{No}}$

Why?



$\boxed{\text{Moral: } \lim\limits_{x \to c} f(x) \text{ doesn't necessarily equal } f(c).}$

Ex 4 Evaluate $\lim_{x \to 0} \frac{1}{x}$ numerically

Recall $\lim_{x \to c} f(x)$. What is $f(x)$? $f(x) = \frac{1}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-10	-100	-1000	$-$	1000	100	10

$-\infty$ ∞

Hence $\lim_{x \to 0} \frac{1}{x} = \text{DNE}$

One-Sided Limits

Definition: a one-sided limit is the value that the function on $f(x) \to L$ as $x \to c$ from the left or right.

Left-sided Limit: If $f(x) \to L$ as $x \to c$ from the left $\lim_{x \to c^-} f(x) = L$.

Right-sided Limit: If $f(x) \to L$ as $x \to c$ from the right $\lim_{x \to c^+} f(x) = L$.

Revist Ex1: Evaluate $\lim_{x\to 4^-}(2x-3)$ and $\lim_{x\to 4^+}(2x-3)$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	4.8	4.98	4.998	–	5.002	5.02	5.2

Left → 5 ← Right

Hence $\lim_{x\to 4^-}(2x-3)=5$, $\lim_{x\to 4^+}(2x-3)=5$

Ex5: Evaluate $\lim_{x\to -1^-}\frac{1}{(x+1)^2}$ and $\lim_{x\to -1^+}\frac{1}{(x+1)^2}$

x	-1.1	-1.01	-1.001	-1	-0.999	-0.99	-0.9
$f(x)$	100	10000	1000000	–	1000000	10000	100

Left → ← Right

∞ ∞

Hence $\lim_{x\to -1^-}\frac{1}{(x+1)^2}=\infty$, $\lim_{x\to -1^+}\frac{1}{(x+1)^2}=\infty$

Revist Ex4: Evaluate $\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0^+} \frac{1}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-10	-100	-1000	$-$	1000	100	10

$\longrightarrow$ Left $-\infty$ $\qquad$ $\longleftarrow$ Right ∞

Hence $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, $\lim_{x \to 0^+} \frac{1}{x} = \infty$

MA 16010 Lesson 2: Finding Limits Graphically

Graphically, we will look at the portion of the curve of $f(x)$ near $x = c$ and see what the function value, y, approaches as x gets closer to c from the left or the right, respectively.

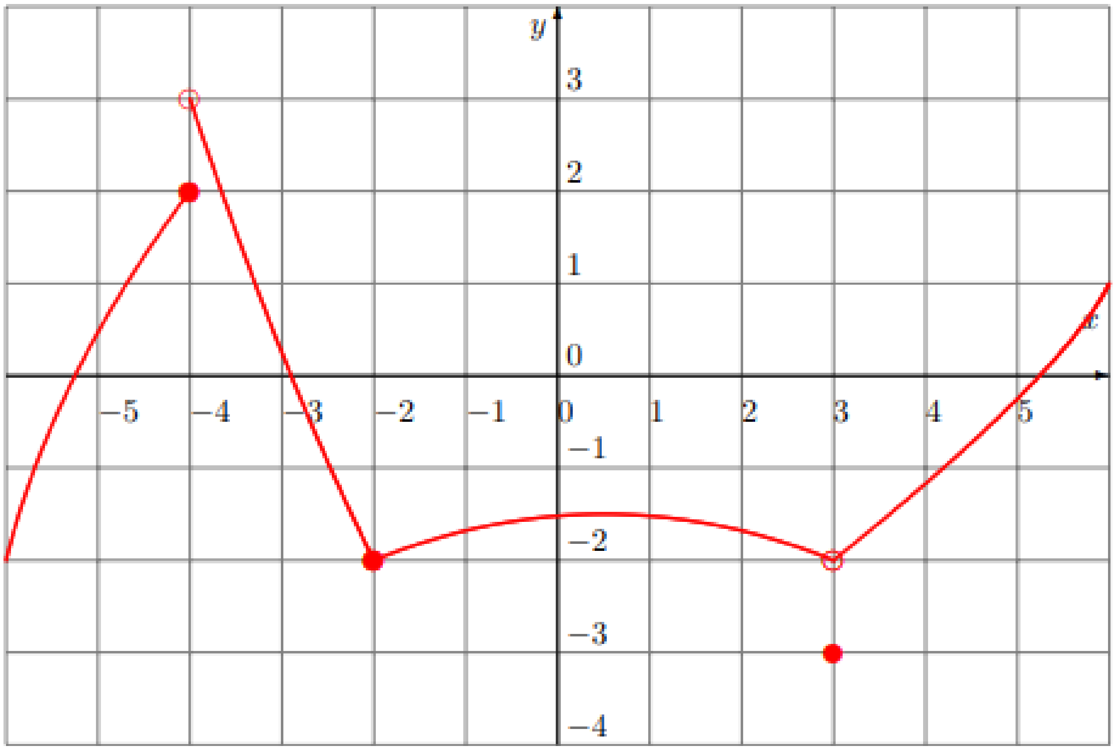
If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$,

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)} \quad (*)$$

Note this doesn't imply that $(*) = f(c)$.

Example 3 (From Worksheet)

3. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x\to -4^-} f(x) = 2$

B) $\lim_{x\to -4^+} f(x) = 3$

C) $\lim_{x\to -4} f(x) =$ DNE

D) $f(-4) = 2$

E) $\lim_{x\to -2^-} f(x) = -2$

F) $\lim_{x\to -2^+} f(x) = -2$

G) $\lim_{x\to -2} f(x) = -2$

H) $f(-2) = -2$

I) $\lim_{x\to 3^-} f(x) = -2$

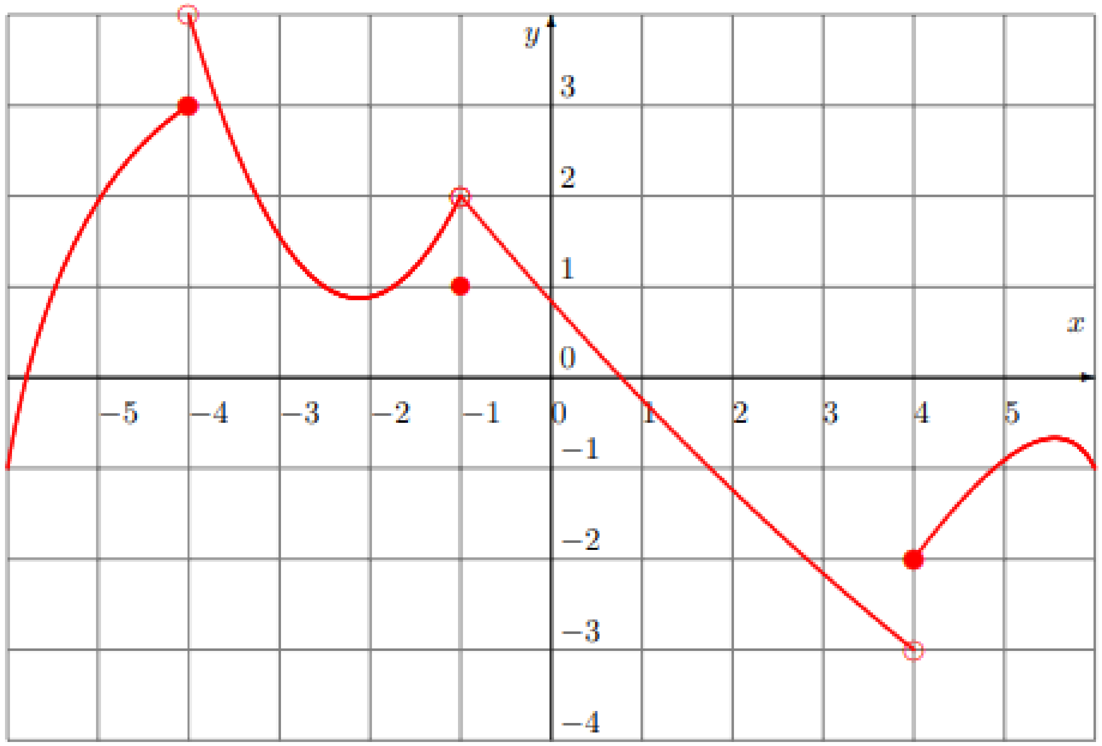
J) $\lim_{x\to 3^+} f(x) = -2$

K) $\lim_{x\to 3} f(x) = -2$

L) $f(3) = -3$

Example 1 (From Worksheet)

1. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x \to -4^-} f(x) =$ 3

E) $\lim_{x \to -1^-} f(x) =$ 2

I) $\lim_{x \to 4^-} f(x) =$ −3

B) $\lim_{x \to -4^+} f(x) =$ 4

F) $\lim_{x \to -1^+} f(x) =$ 2

J) $\lim_{x \to 4^+} f(x) =$ −2

C) $\lim_{x \to -4} f(x) =$ DNE

G) $\lim_{x \to -1} f(x) =$ 2

K) $\lim_{x \to 4} f(x) =$ DNE

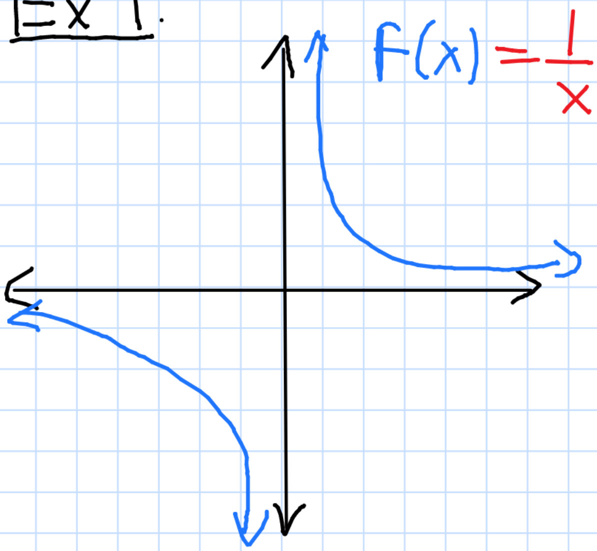
D) $f(-4) =$ 3

H) $f(-1) =$ 1

L) $f(4) =$ −2

Lesson 2: Finding Limits Graphically

Ex 1:



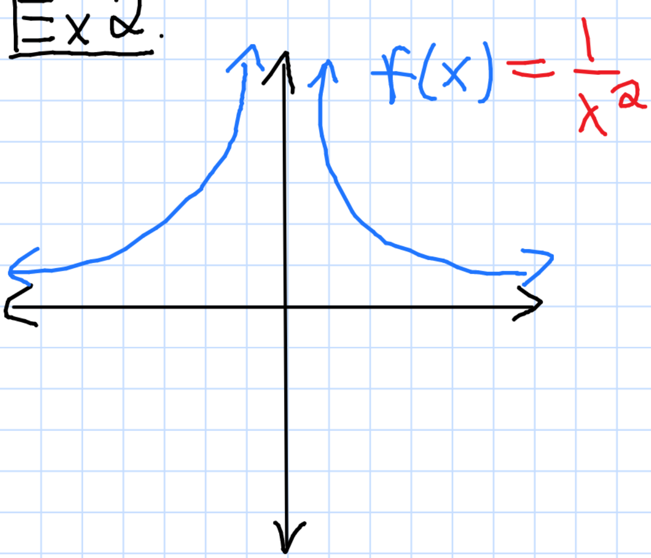
ⓐ $\lim_{x \to 0^-} f(x) = -\infty$

ⓑ $\lim_{x \to 0^+} f(x) = \infty$

ⓒ $\lim_{x \to 0} f(x) =$ DNE

ⓓ $f(0) =$ undefined

Ex 2:



ⓐ $\lim_{x \to 0^-} f(x) = \infty$

ⓑ $\lim_{x \to 0^+} f(x) = \infty$

ⓒ $\lim_{x \to 0} f(x) = \infty$

ⓓ $f(0) =$ undefined