

Lesson 34: Exponential Growth

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Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Try getting all y terms to one side.

$$\frac{dy}{y} = 2dt$$

$$\int \frac{dy}{y} = \int 2dt$$

$$e \ln y = 2t + C$$

$$y = e^{2t+C}$$

$$= e^{2t} \cdot e^C$$

$$= C e^{2t}$$

$e^C \rightarrow$ constant

Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Now with $y = C e^{2t}$ use

$$100 = C e^{2(0)} = C$$

So $y = 100 e^{2t}$

Exponential Growth Model

If y is a differential function of t such that

$$\frac{dy}{dt} = y' = ky \text{ for some constant } k$$

then $y = Ce^{kt}$ where C is a constant.

k - proportionality constant or growth rate

C - the initial value of y

If $k > 0$ and $C > 0$, then this model is called the exponential growth model.

Ex 2: The rate of change of a population P is proportional to P (use k for the proportionality constant). Answer the following questions.

(a) What is dP/dt ?

$$\frac{dP}{dt} = kP$$

(b) Find P .

$$P = Ce^{kt}$$

(c) If $P = 200$ when $t = 0$ and $P = 400$ when $t = 2$, what is $P(4)$?

$$200 = Ce^{k(0)} = C$$

$$P = 200e^{kt}$$

$$400 = 200e^{2k}$$

$$2 = e^{2k}$$

$$\ln(2) = 2k$$

$$k = \frac{\ln(2)}{2}$$

$$P = 200 \exp\left[\frac{\ln(2)}{2} t\right]$$

$$P(4) = 200 \exp\left[\frac{\ln(2)}{2} \cdot 4\right]$$

$$\begin{aligned}
 &= 200 \exp[2 \ln 2] \\
 &= 200 \exp[\ln 2^2] \\
 &= 200 \cdot 4 = \boxed{800}
 \end{aligned}$$

(d) If $P = 200$ when $t = 1$ and $P = 400$ when $t = 2$, what is $P(4)$?

$$200 = C e^k \quad \textcircled{1}$$

$$400 = C e^{2k} = \underbrace{C e^k}_{\textcircled{1}} e^k \quad \textcircled{2}$$

Plug $\textcircled{1}$ into $\textcircled{2}$

$$400 = 200 e^k$$

$$2 = e^k$$

$$\ln 2 = k$$

Plug k into $\textcircled{1}$.

$$200 = C e^{\ln 2} = 2C$$

$$C = 100$$

$$P = 100 \exp[\ln(2)t]$$

$$P(4) = 100 \exp[\ln(2) \cdot 4]$$

$$= 100 \exp[\ln 2^4]$$

$$= 100 (16) = \boxed{1600}$$

Ex 3: In a savings account where the interest is compounded continuously, if the initial investment is 500 dollars and the annual interest rate is 3%, how much money will there be in 10 years?

Note $C = 500 > 0$ and $k = 0.03 > 0$. So

$$y = 500 \exp[0.03t]$$

$$\begin{aligned}
 y(10) &= 500 \exp[0.03(10)] \\
 &= 500 \exp[0.3] \\
 &\approx \boxed{\$674.93}
 \end{aligned}$$

Ex 3 (Continuation) How long does it take to double the initial investment?

Previously we found $y = 500 \exp[0.03t]$. . .
 Double the initial investment $\Rightarrow 2(500)$. So

$$\begin{aligned}
 2(500) &= 500 \exp[0.03t] \\
 \ln 2 &= 0.03t \\
 t &= \frac{\ln 2}{0.03} \approx \boxed{23.1 \text{ years}}
 \end{aligned}$$

Ex 4: In a savings account where the interest is compounded continuously, if the initial investment is 100 dollars and there are 150 dollars in 8 years, what is the annual interest rate?

Solving for k.

$$\left. \begin{array}{l} k - \text{interest rate} > 0 \\ C = 100 > 0 \end{array} \right\} y = 100 e^{kt}$$

The question also states $y(8) = 150$

$$\begin{array}{l|l}
 150 = 100 e^{k(8)} & \ln(3/2) = 8k \\
 \frac{3}{2} = \frac{150}{100} = e^{8k} & k = \frac{1}{8} \ln\left(\frac{3}{2}\right) \approx 0.05 \\
 & \Rightarrow \boxed{5\%}
 \end{array}$$