Friday, January 16, 2026 1:15 PM

Lesson 3: Finding Limits Analytically

There are 3 different cases to consider.

① $f(c)$ returns a # (it could be 0)

i.e. $f(x)$ is continuous @ $x=c$

i.e. $\lim_{x \to c} f(x) = f(c)$

Ex 1: $\lim_{x \to 4} (2x-3) = 2(4) - 3 = 8 - 3 = 5$

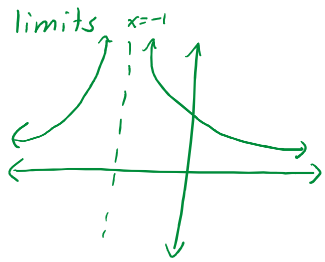
② $f(c)$ returns $\frac{\text{nonzero } \#}{0}$

i.e. Vertical Asymptote @ $x=c$

i.e. $\lim_{x \to c} f(x) = \pm\infty$ or DNE

Ex 2: $\lim_{x \to -1} \frac{1}{(x+1)^2}$

$f(-1) = \frac{1}{(-1+1)^2} = \frac{1}{0} \Rightarrow$ We need to check the left and right limits



$\left.\begin{aligned} \lim_{x \to -1^-} \frac{1}{(x+1)^2} &= \infty \\ \lim_{x \to -1^+} \frac{1}{(x+1)^2} &= \infty \end{aligned}\right\} \Rightarrow \lim_{x \to -1} \frac{1}{(x+1)^2} = \infty$

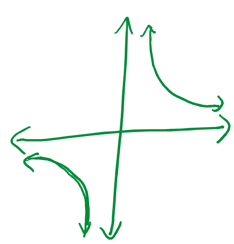
Ex 3: $\lim_{x \to -1} \frac{-1}{(x+1)^2} = -\left[\lim_{x \to -1} \frac{1}{(x+1)^2}\right] = -\infty$ by Ex 2

Ex 4: $\lim_{x \to 0} \frac{1}{x}$

$f(0) = \frac{1}{0} \Rightarrow$ We need to check the left and right limits

$\left.\begin{aligned} \lim_{x \to 0^-} \frac{1}{x} &= -\infty \\ \lim_{x \to 0^+} \frac{1}{x} &= \infty \end{aligned}\right\}$ BUT they don't match

$\lim_{x \to 0} \frac{1}{x} = \text{DNE}$



③ $f(c)$ returns $\frac{0}{0}$

i.e. $f(x)$ has a hole (if a factor cancels out) or VA @ $x=c$ (no factor cancels)

(3) $f(c)$ returns $\frac{-}{0}$

i.e. $f(x)$ has a hole (if a factor cancels out) or VA @ $x = c$ (no factor cancels)

Ex 5: $\lim\limits_{x \to 3} \frac{x^3 - 3x^2}{x-3}$

$f(3) = \frac{3^3 - 3 \cdot 3^2}{3-3} = \frac{27-27}{3-3} = \frac{0}{0} \Rightarrow$ Factor!

$\lim\limits_{x \to 3} \frac{x^3 - 3x^2}{x-3} = \lim\limits_{x \to 3} \frac{x^2\cancel{(x-3)}}{\cancel{x-3}} = \lim\limits_{x \to 3} x^2 = 3^2 = 9$

Ex 6: $\lim\limits_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

$f(4) = \frac{\sqrt{4} - 2}{4-4} = \frac{2-2}{4-4} = \frac{0}{0} \Rightarrow$ Factor!

$= \lim\limits_{x \to 4} \frac{\cancel{\sqrt{x} - 2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}}$

$= \lim\limits_{x \to 4} \frac{1}{\sqrt{x}+2}$

$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

Difference of Squares

$a^2 - b^2 = (a+b)(a-b)$

$x - 4 = x - 2^2$

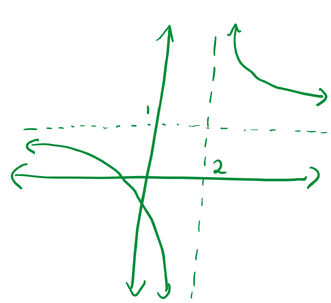
$= (\sqrt{x})^2 - 2^2$

$= (\sqrt{x}+2)(\sqrt{x}-2)$

Ex 7: $\lim\limits_{x \to 2} \frac{x^2 - 2x}{(x-2)^2} = \lim\limits_{x \to 2} \frac{x\cancel{(x-2)}}{(x-2)^{\cancel{2}}} = \lim\limits_{x \to 2} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0}$

But now it looks like Case 2. So we need to check the left and right limits.

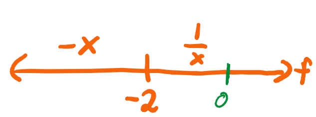
$\frac{x}{x-2} = \frac{x-2+2}{x-2} = \frac{x-2}{x-2} + \frac{2}{x-2} = 1 + \frac{2}{x-2}$



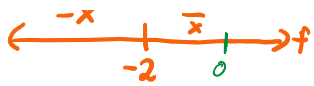
By the graph,

$\lim\limits_{x \to 2} \frac{x^2-2x}{(x-2)^2} = \lim\limits_{x \to 2} \frac{x}{x-2} = \text{DNE}$

Ex 8: $f(x) = \begin{cases} 1/x & \text{if } x \geq -2 \\ -x & \text{if } x < -2 \end{cases}$



$\left(-x \quad \text{if } x < -2 \right.$



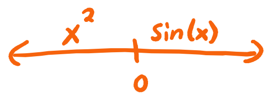
ⓐ $\lim_{x \to -2} f(x)$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} -x = -(-2) = 2$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} \frac{1}{x} = \frac{1}{-2}$ ⨯ DNE

ⓑ $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x} = \text{DNE}$

Ex 9: $f(x) = \begin{cases} \sin(x) & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$



Find $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$ ✓ $\lim_{x \to 0} f(x) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(x) = \sin(0) = 0$