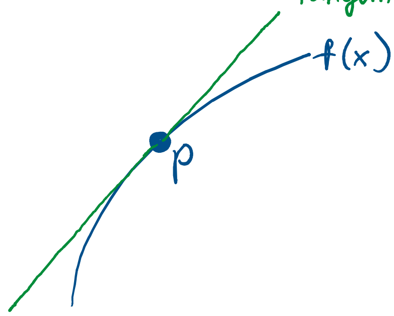
Lesson 5: The Derivative

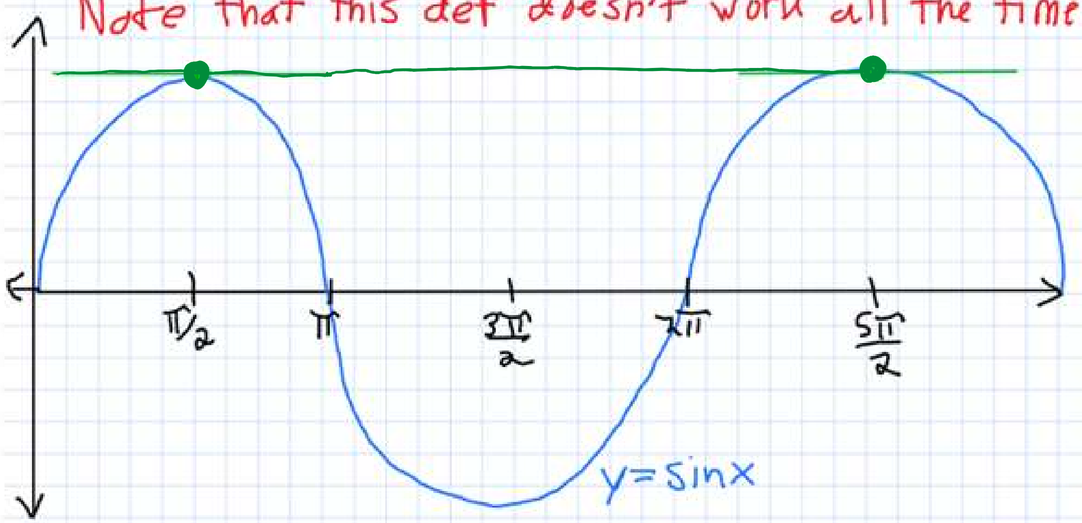
Tangent lines are Important in Calculus!!!

What is a Tangent line?



Tangent Line is Graph of f(x) at a point P that is straight line that touches f(x) at point P, but does not cross f(x).

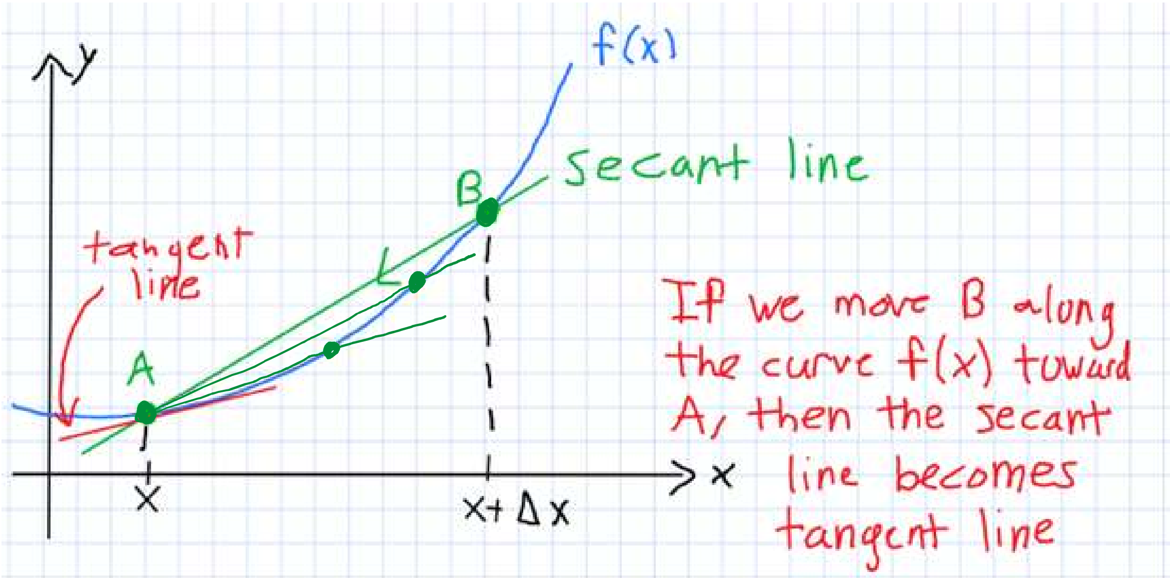
Note that this def doesn't work all the time

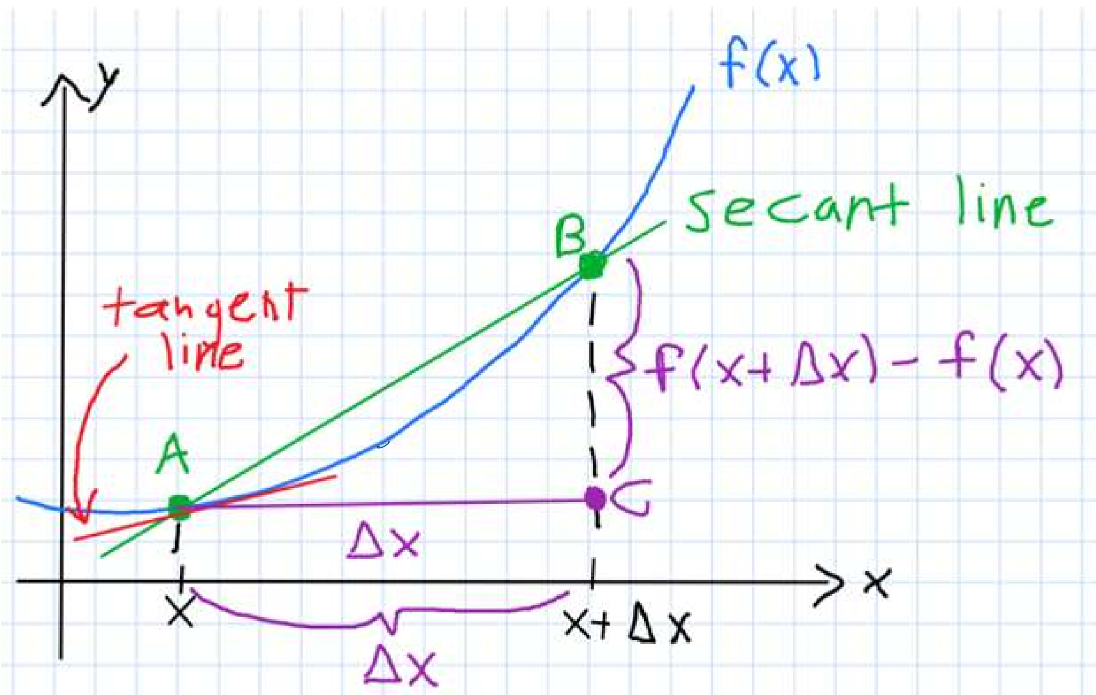


B/c look @ how those green lines cross

We need a precise definition for all scenarios. Before we do so, let's recap secant lines.

A secant line of f(x) is a straight line that goes through 2 distinct points on f(x).





If we want to find the tangent line to $f(x)$ at a specific point, we can do that with the secant.

How can we achieve that?

If we knew slope of tangent line, then we can use point-slope formula to find the eqn of tangent line.

Recall point-slope formula is
Given m-slope and (x_1, y_1)-point: $y - y_1 = m(x - x_1)$

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$$\text{Slope of Tangent line} = \lim_{\Delta x \to 0} \text{Slope of secant line} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

also known as the difference quotient

How is the slope of the tangent line related to the derivative?
They are the same!

Def: The derivative of $f(x)$ at x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{where } h = \Delta x$$

and provided that the limit exists.

Different Notations: y', $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}[f(x)]$

Game Plan: To find derivative using the limit definition of derivative, follow the following steps:

ex. $f(\square) = \square^2 + \square$

1. Find $f(x+h)$
2. Find $-f(x)$
3. Find $f(x+h) - f(x)$ by adding 1 + 2
4. Find $\frac{f(x+h) - f(x)}{h}$ by diving 3 by h
5. Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by taking $\lim_{h \to 0}$ of 4

Ex 1: Find the derivative of $f(x) = x + 5$ using the limit definition.

Step 1: $f(x+h)$

$f(x) = x + 5$

$f(x+h) = (x+h) + 5 = x + h + 5$

Step 2: $-f(x)$

$f(x) = x + 5$

Step 3: $f(x+h) - f(x)$

$$\frac{\cancel{x} + h + \cancel{5} + \cancel{-x} \cancel{-5}}{h}$$

$f(x+h) - f(x) = h$

Step 2: $-f(x)$

$f(x) = x + 5$

$-f(x) = -x - 5$

$f(x+h) - f(x) = h$

Step 4: $\dfrac{f(x+h) - f(x)}{h} \overset{\text{by } ③}{=} \dfrac{h}{h} = 1$

Step 5: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \overset{\text{by } ④}{=} \lim\limits_{h \to 0} (1) = 1 = f'(x)$

Useful Formulas:

- $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- $a^2 - b^2 = (a-b)(a+b)$

Ex 2: Given $f(x) = x^2 - 3$.

ⓐ Find the slope of the tangent line. $\boxed{\text{i.e. } f'(x)}$

Step 1: $f(x+h) = (x+h)^2 - 3 = \cancel{x^2} + 2xh + h^2 \cancel{-3}$

Step 2: $-f(x) = -[x^2 - 3] = \cancel{-x^2} \cancel{+3}$

Step 3: $f(x+h) - f(x)$. Add ①+② $= 2xh + h^2$

Step 4: Divide ③ by h to get

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + h$$

Step 5: Take the $\lim\limits_{h \to 0}$ of ④

$$\lim_{h \to 0} (2x + h) = 2x + 0 = 2x = f'(x)$$

Game Plan: To find the eqn of the tangent line to $f(x)$ at the point $x = c$, follow the following steps:

① Find $f'(x)$.

② Calculate $f'(c)$

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③ Calculate $f(c)$
④ Plug ② + ③ into the point-slope formula

$$y - f(c) = m(x-c) \qquad m = f'(c) \text{ always}$$
$$y - f(c) = f'(c)(x-c)$$

Ex 2: Given $f(x) = x^2 - 3$.

ⓑ Find the equation of the tangent line to $f(x)$ at $x=2$.

Step 1: Find $f'(x)$.

By Part a, $f'(x) = 2x$

Step 2: Find $f'(2)$.

$f'(2) = 2(2) = 4$

Step 3: Find $f(2) = 2^2 - 3 = 4 - 3 = 1$

Step 4: Point-Slope Formula

$$y - f(2) = f'(2)(x-2)$$
$$y - 1 = 4(x-2)$$
$$y - 1 = 4x - 8$$
$$\underline{\quad +1 \qquad\quad +1 \quad}$$
$$y = 4x - 7$$

Ex 3: Determine what $f'(x)$ is given

$$\lim_{h \to 0} \frac{(2+h)^3 + (2+h)^2 - 12}{h} \text{ ?}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Let $x = 2$:

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2)$$

$$f(2+h) = (2+h)^3 + (2+h)^2$$

Let $x = 2+h$

$f(2+h)=(2+h)^3+(2+h)^2$

Let $x=2+h$

$$f(x)=x^3+x^2$$

Check that $f(2)$ is actually 12.

$$f(2)=2^3+2^2=8+4=12$$

I know $f(x)=x^3+x^2$. So $f'(x)=3x^2+2x$

Why? In a future lesson I say why