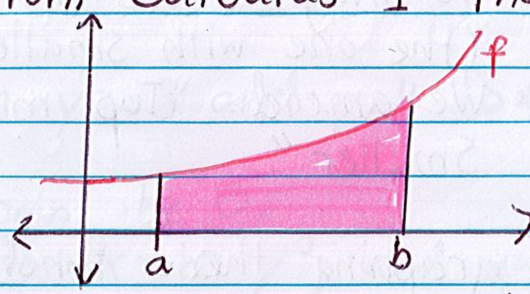
Lesson 13: Area Between Two Curves II

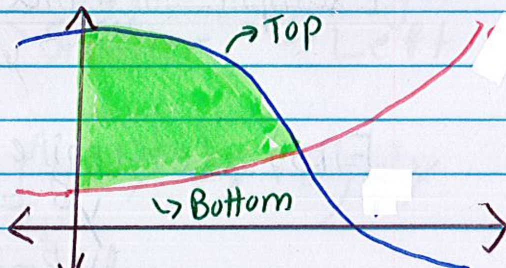
Last time, we recalled from Calculus I that

$$\int_{x=a}^{x=b} f(x)\,dx \implies$$



and with that interpretation found a formula for the area between two curves with respect to x, i.e.

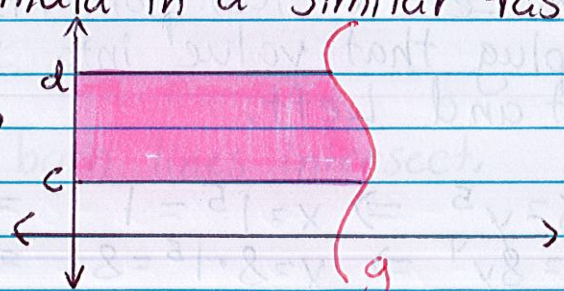
$$\text{Area} = \int_{x=a}^{x=b} (\text{Top} - \text{Bottom})\,dx$$



Today's lesson, we will be focussing on the area between two curves with respect to y.

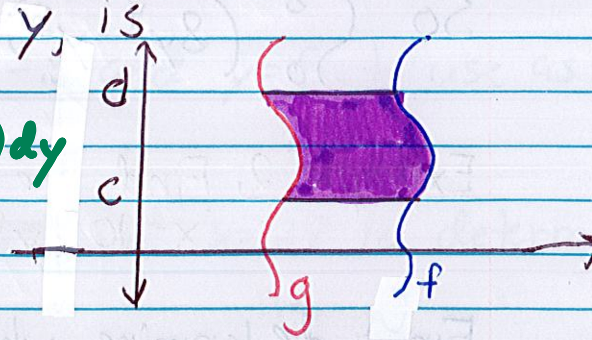
We will define the formula in a similar fashion. So

$$\int_{y=c}^{y=d} g(y)\,dy \implies$$



Hence we can say the formula for the area between two curves with respect to y, is

$$\text{Area} = \int_{y=c}^{y=d} (\textbf{Bigger} - \textbf{Smaller})\,dy$$



So what has changed?

- The roles of x and y switch.
- Given two curves which are in terms of y, we want to integrate (the one with larger x-values) mines (the one with smaller x-values).
- We amend "Top minus Bottom" to "Bigger minus Smaller"

Now graphing these functions can be quiet difficult so I'll be introducing a new way of doing this problems.

Example 1: Find the area of the region bounded by $x = y^5$ and $x = 8y^4$

First determine where both lines intersect.

$$y^5 = x = 8y^4$$
$$y^5 - 8y^4 = 0$$
$$y^4(y-8) = 0$$
$$y = 0, 8$$

Next choose a # between $y=0$, and $y=8$ to use as a test point.

ex. Let the test point be 1.

Now plug that value into $x = y^5$, and $x = 8y^4$ to determine **Bigger** and **Smaller**.

$x = y^5 \Rightarrow x = 1^5 = 1 \Rightarrow$ Smaller
$x = 8y^4 \Rightarrow x = 8 \cdot 1^5 = 8 \Rightarrow$ Bigger

So $\int_0^8 (8y^4 - y^5)\,dy = \left(\frac{8y^5}{5} - \frac{y^6}{6}\right)\Big]_0^8 = \frac{131072}{15}$

Example 2: Find the area of the region bounded by $x = 10 - y^2$, and $x = y - 2$

First determine where both lines intersect.

$$10-y^2 = y-2$$
$$0 = y^2+y-12$$
$$0 = (y+4)(y-3)$$
$$y = -4,\ 3$$

Next choose a # between $y=-4$ and $y=3$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x=10-y^2$, and $x=y-2$ to determine **Bigger** and **Smaller**

$$x=10-y^2 \Rightarrow x=10-0^2=10 \Rightarrow \text{Bigger}$$
$$x=y-2 \Rightarrow x=0-2=-2 \Rightarrow \text{Smaller}$$

So
$$\int_{-4}^{3}\left(10-y^2-(y-2)\right)dy = \int_{-4}^{3}\left(10-y^2-y+2\right)dy$$
$$= \int_{-4}^{3}\left(-y^2-y+12\right)dy$$
$$= \left(-\frac{y^3}{3}-\frac{y^2}{2}+12y\right)\Big]_{-4}^{3} = \frac{343}{6}$$

Example 3: Find the area of the region bounded by $x=y^2+2$ and $x=27$

First determine where both lines intersect.
$$y^2+2 = 27$$
$$y^2 = 25$$
$$y = \pm 5$$

Next choose a # between $y=-5$ and $y=5$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x=y^2+2$ and $x=27$ to determine **Bigger** and **Smaller**

$x = y^2 + 2 \Rightarrow x = 0^2 + 2 = 2 \Rightarrow$ Smaller
$x = 27 \Rightarrow x = 27 \Rightarrow$ Bigger

$$\text{So } \int_{-5}^{5} (27 - (y^2+2))\,dy = \int_{-5}^{5} (27 - y^2 - 2)\,dy$$
$$= \int_{-5}^{5} (25 - y^2)\,dy$$
$$= \left(25y - \frac{y^3}{3}\right)\Big]_{-5}^{5} = \frac{500}{3}$$

Example 4: Find the area of the region bounded by $x = y^2 - 8$ and $x = 4 - y^2$

First determine where both lines intersect

$$y^2 - 8 = 4 - y^2$$
$$2y^2 = 12$$
$$y^2 = 6$$
$$y = \pm\sqrt{6}$$

Next choose a # between $y = -\sqrt{6}$ and $y = \sqrt{6}$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x = y^2 - 8$ and $x = 4 - y^2$ to determine **Bigger** and **Smaller**

$x = y^2 - 8 \Rightarrow x = 0^2 - 8 = -8 \Rightarrow$ Smaller
$x = 4 - y^2 \Rightarrow x = 4 - 0^2 = 4 \Rightarrow$ Bigger

$$\text{So } \int_{-\sqrt{6}}^{\sqrt{6}} (4 - y^2 - (y^2 - 8))\,dy = \int_{-\sqrt{6}}^{\sqrt{6}} (4 - y^2 - y^2 + 8)\,dy$$
$$= \int_{-\sqrt{6}}^{\sqrt{6}} (12 - 2y^2)\,dy$$
$$= \left(12y - \frac{2y^3}{3}\right)\Big]_{-\sqrt{6}}^{\sqrt{6}} = 16\sqrt{6}$$