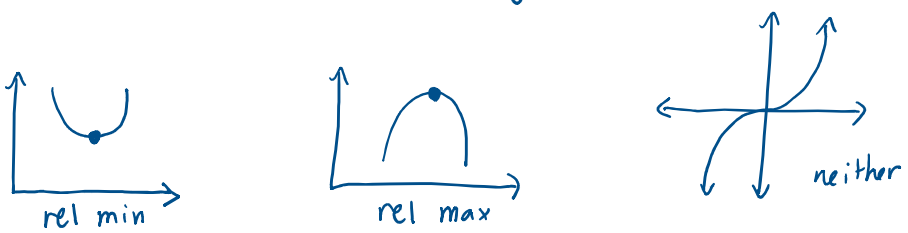
Monday, April 13, 2026 12:15 PM

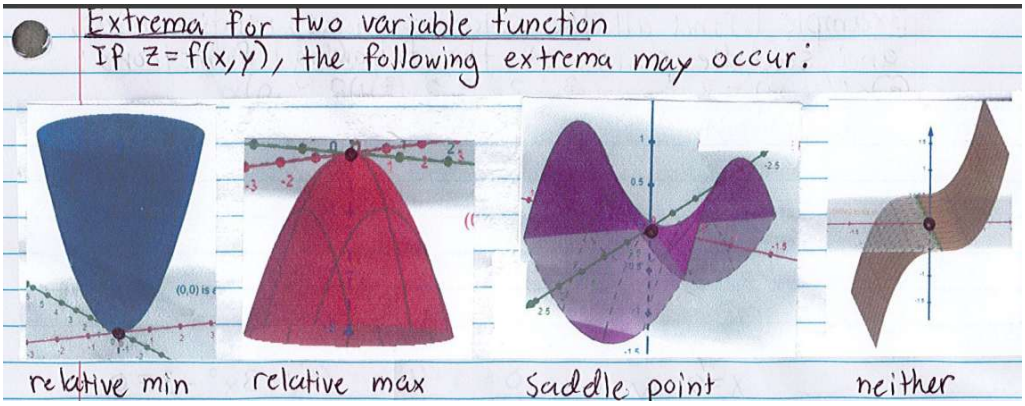
Lesson 29: Extrema of Functions of Two Variables

Recall from Calculus 1, when $y = f(x)$

- x_0 is a critical number if $f'(x_0) = 0$
- By the Second Derivative Test,
  - ↳ if $f''(x_0) > 0 \Rightarrow$ relative min @ $(x_0, f(x_0))$
  - ↳ if $f''(x_0) < 0 \Rightarrow$ relative max @ $(x_0, f(x_0))$
  - ↳ if $f''(x_0) = 0 \Rightarrow$ test is inconclusive

Typically we apply the First Derivative Test to gain more information





We see that in all cases of extrema, the tangent plane to the graph is parallel to the xy-plane. We can describe this as

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

Finding extrema of functions of two variables

Let $z = f(x,y)$.

① Find all the critical points.

i.e. All (x_0, y_0) such that $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

② Compute f_{xx}, f_{xy}, f_{yy} and

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

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$$D = f_{xx} f_{yy} - (f_{xy})^2$$

where D is known as the discriminant.

③ For every given critical pt (x_0, y_0), evaluate D and f_{xx} at (x_0, y_0)

④ Apply the Second Derivative Test for multivariable functions.

(i) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$ $\Rightarrow$ rel min

(ii) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$ $\Rightarrow$ rel max

(iii) If $D(x_0,y_0) < 0$ $\Rightarrow$ Saddle pt

(iv) If $D(x_0,y_0) = 0$ $\Rightarrow$ Test is inconclusive

Ex1:

Given the table below,

(a,b)	$f_{xx}(a,b)$	$f_{yy}(a,b)$	$f_{xy}(a,b)$	D	
$(9,4)$	-1	-1	-1	$(-1)(-1)-(-1)^2 = 1-1 = 0$	Inconclusive
$(-2,2)$	4	3	-4	$4\cdot 3-(-4)^2 = 12-16$	Saddle pt
$(4,5)$	8	5	6	$8\cdot 5-6^2 = 40-36 > 0$	rel min

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

Ex2: Find all the relative max or min and saddle points of the following functions.

(a) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

① Critical pts:

$$f_x = 6xy + 0 - 6x - 0 + 0 = 6xy - 6x = 0 \quad (I)$$

$$f_y = 3x^2 + 3y^2 - 0 - 6y + 0 = 3x^2 + 3y^2 - 6y = 0 \quad (II)$$

Solve (I)

$6xy - 6x = 0$

$6x(y-1) = 0$

$x = 0$ or $y = 1$ ✓

Plug $x = 0$ in (II)

$0 + 3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0, y = 2$

Critical pts: $(0,0), (0,2)$

Plug $y = 1$ in (II)

$3x^2 + 3 - 6 = 0$

$3x^2 - 3 = 0$

$3(x^2-1) = 0$

$x = \pm 1$

Critical Pts: $(1,1), (-1,1)$

② Compute f_{xx}, f_{xy}, f_{yy}, and D.

$f_x = 6xy - 6x$ $\quad f_y = 3x^2 + 3y^2 - 6y$

$f_{xx} = 6y - 6$ $\quad f_{yy} = 6y - 6$

$f_{xy} = 6x$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$f_{xx} = 6y - 6$ $\quad f_{yy} = 6y - 6$

$f_{xy} = 6x$

	$f_{xx} = 6y-6$	$f_{yy} = 6y-6$	$f_{xy} = 6x$	D	
$(0,0)$	-6	-6	0	$36-0^2$	rel max
$(0,2)$	6	6	0	$36-0^2$	rel min
$(1,1)$	0	0	6	$0-6^2=-36$	saddle pt
$(-1,1)$	0	0	-6	$0-(-6)^2$ $=-36$	saddle pt

(b) $f(x,y) = \frac{2}{3}y^3 + x^2 - 4yx - 10y + 6$

① Critical Point(s)

$f_x = 0 + 2x - 4y + 0 + 0 = 2x - 4y = 0$ (I)

$f_y = 2y^2 + 0 - 4x - 10 + 0 = 2y^2 - 4x - 10 = 0$ (II)

Solve (I).

$2x - 4y = 0$

$2x = 4y$

$x = 2y$

Plug $x = 2y$ into (II).

$2y^2 - 4(2y) - 10 = 0$

$2y^2 - 8y - 10 = 0$

$2(y^2 - 4y - 5) = 0$

$2(y-5)(y+1) = 0$

$y = 5, -1$

Plug $y = 5, -1$ into $x = 2y$

When $y = 5$, $x = 2 \cdot 5 = 10$

CP: $(10, 5)$

When $y = -1$, $x = 2 \cdot (-1) = -2$

CP: $(-2, -1)$

② Compute f_{xx}, f_{xy}, f_{yy}, and D.

$f_x = 2x - 4y$ $\quad f_y = 2y^2 - 4x - 10$

$f_{xx} = 2$ $\quad f_{yy} = 4y$

$f_{xy} = -4$

$D = f_{xx} f_{yy} - (f_{xy})^2$

	$f_{xx} = 2$	$f_{yy} = 4y$	$f_{xy} = -4$	D	Conclusion
$(10,5)$	2	20	-4	$40-(-4)^2$	rel min
$(-2,-1)$	2	-4	-4	$-8-(-4)^2$	saddle pt

(c) $f(x,y) = \frac{3}{2}x^4 - yx^2 + 20x^2 + \frac{1}{2}y^2 - 3$

① Critical Pts:

$f_x = 6x^3 - 2xy + 40x + 0 + 0 = 6x^3 - 2xy + 40x = 0$ (I)

$f_y = 0 - x^2 + 0 + y + 0 = -x^2 + y = 0$ (II)

Solve (II)

$-x^2 + y = 0$

Plug $y = x^2$ into (I)

$6x^3 - 2x(x^2) + 40x = 0$

Plug $x = 0$ into $y = x^2$

$(0, 0)$

Solve (II)

$-x^2 + y = 0$

$y = x^2$

Plug $y = x^2$ into (I)

$6x^3 - 2x(x^2) + 40x = 0$

$6x^3 - 2x^3 + 40x = 0$

$4x^3 + 40x = 0$

$4x(x^2 + 10) = 0$

$x = 0$ → No solution

$(0, 0)$

only critical pt.

② Compute f_{xx}, f_{xy}, f_{yy}, and D

$f_x = 6x^3 - 2xy + 40x$

$f_{xx} = 18x^2 - 2y + 40$

$f_{xy} = 0 - 2x + 0$

$f_y = -x^2 + y$

$f_{yy} = 0 + 1 = 1$

	$f_{xx} = 18x^2 - 2y + 40$	$f_{yy} = 1$	$f_{xy} = -2x$	D	
$(0,0)$	40	1	0	1	rel min