Lesson 3: Integration By Substitution I

Warm-Up: Determine the Inner/outer Functions for the following:

ⓐ $h(x) = \sin^3(x) = [\sin(x)]^3$

Outer function = x^3

Inner function = $\sin(x)$

Check: $h(x) = f(g(x)) = f(\sin(x))$
$= (\sin(x))^3$ ✓

ⓑ $h(x) = \sqrt{3x+2}$

Outer function = $\sqrt{x}$

Inner function = $3x+2$

Check: $h(x) = f(g(x)) = f(3x+2)$
$= \sqrt{3x+2}$

ⓒ $h(x) = \tan(3x^2)$

Outer function = $\tan(x)$

Inner function = $3x^2$

Check: $h(x) = f(g(x))$
$= f(3x^2)$
$= \tan(3x^2)$

Integration By Substitution is kinda of the Integration Version of the Chain Rule. Also referred to as "Change of Variables"

Ex 1: Find $\int \underline{2x}\cos(x^2)\,dx$.

Idea to solve Ex 1 is to undo the chain rule.

First determine if you have a function within a function.

In this example, $\cos(x^2)$
where $\cos(x)$ is outer function
and x^2 is inner function.

What's the derivative of the inner function?

$$x^2 \rightarrow 2x$$

Do you see $2x$ in the integrand? Yes!

Do you see 2x in the integ...

Let's recall chain rule: $y' = f'(g(x)) \cdot g'(x)$ for $y = f(g(x))$

So from our example: $y' = \cos(x^2) \cdot 2x$

i.e. $f(x) = \ ?$ $\qquad g(x) = x^2$

$f'(x) = \cos(x)$ $\qquad g'(x) = 2x$

So what's $f(x)$?

$$f(x) = \int f'(x)\,dx = \int \cos(x)\,dx = \sin(x)$$

Hence $y = f(g(x)) = \sin(x^2)$

i.e. $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

In practice, we fast track this method by doing a change of variable using the inner function

Solution to Ex 1 (faster solution)

$$\int \cos(\underbrace{x^2}_{u}) \cdot \underbrace{2x\,dx}_{du} \xrightarrow[\substack{\frac{du}{dx} = 2x \\ du = 2x\,dx}]{u = x^2} \int \cos(u)\,du = \sin(u) + C$$

$$= \sin(x^2) + C$$

Note we may not see what du equals in our integrand. So you may have to do some equation manipulation.

Ex 2: Compute

(a) $$\int \sqrt{4x+1}\,\underbrace{dx} \xrightarrow[\substack{du = 4\underbrace{dx} \\ \frac{du}{4} = dx}]{u = 4x+1} \int \sqrt{u}\,\frac{du}{4} = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2}$$

$$= \frac{1}{6}(4x+1)^{3/2} + C$$

(b) $\int 4e^{2x}\,dx$ $\quad \dfrac{u = 2x}{du = 2\,dx}$ $\quad \dfrac{du}{2} = dx$

$$\int 4e^{u}\,\frac{du}{2} = \int 2e^{u}\,du = 2e^{u}$$

$$= 2e^{2x} + C$$

(c) $\int 3x^{5}e^{x^{6}}\,dx$ $\quad \dfrac{u = x^{6}}{du = 6x^{5}\,dx}$ $\quad \dfrac{du}{6x^{5}} = dx$

$$\int 3\cancel{x^{5}}\,e^{u}\,\frac{du}{6\cancel{x^{5}}} = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u}$$

$$= \frac{1}{2}e^{x^{6}} + C$$

(d) $\displaystyle\int \frac{3\sin(x)}{\cos^{8}(x)}\,dx = \int \frac{3\sin(x)}{(\cos x)^{8}}\,dx = \int 3\sin(x)[\cos(x)]^{-8}\,dx$

$\dfrac{u = \cos(x)}{du = -\sin(x)\,dx}$ $\quad \dfrac{du}{-\sin(x)} = dx$

$$\int 3\cancel{\sin(x)}\cdot u^{-8}\,\frac{du}{-\cancel{\sin(x)}} = -3\int u^{-8}\,du$$

$$= -3\,\frac{u^{-8+1}}{-8+1}$$

$$= -3\,\frac{u^{-7}}{-7}$$

$$= \frac{3}{7}\cdot\frac{1}{u^{7}}$$

$$= \frac{3}{7}\cdot\frac{1}{(\cos(x))^{7}} + C$$

(e) $\int 5e^{\tan(14x)}\sec^{2}(14x)\,dx$ $\quad \dfrac{u = \tan(14x)}{du = \sec^{2}(14x)\cdot 14\,dx}$ $\quad \dfrac{du}{14\sec^{2}(14x)} = dx$

$$\int 5e^{u}\,\cancel{\sec^{2}(14x)}\cdot\frac{du}{14\cancel{\sec^{2}(14x)}}$$

$$= \frac{5}{14}\int e^{u}\,du = \frac{5}{14}e^{u} = \frac{5}{14}e^{\tan(14x)} + C$$

Example 3: Find the function $f(x)$ whose tangent line has the slope $\frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$ for any $x \neq 0$ and whose graph passes through the point $(9, 5/3)$.

$$f'(x) = \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}} \Rightarrow f(x) = \int \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}\,dx$$

$$\int \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}\,dx$$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}x^{-1/2}dx$

$du = \frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,du = dx$

$$\int \frac{u^{1/2}}{4\cancel{\sqrt{x}}} \cdot 2\cancel{\sqrt{x}}\,du = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{2} \cdot \frac{2}{3}\, u^{3/2}$$

$$= \frac{1}{3}(1+\sqrt{x})^{3/2} + C = f(x)$$

Next find C w/ point $(9, 5/3)$.

$$\frac{5}{3} = \frac{1}{3}(1+\sqrt{9})^{3/2} + C$$

$$\frac{5}{3} = \frac{1}{3}(4)^{3/2} + C$$

$$\frac{5}{3} = \frac{1}{3} \cdot 2^3 + C$$

$$\frac{5}{3} = \frac{8}{3} + C$$

$$-\frac{3}{3} = C$$

$$-1 = C$$

$$f(x) = \frac{1}{3}(1+\sqrt{x})^{3/2} + C$$

$$= \frac{1}{3}(1+\sqrt{x})^{3/2} - 1$$