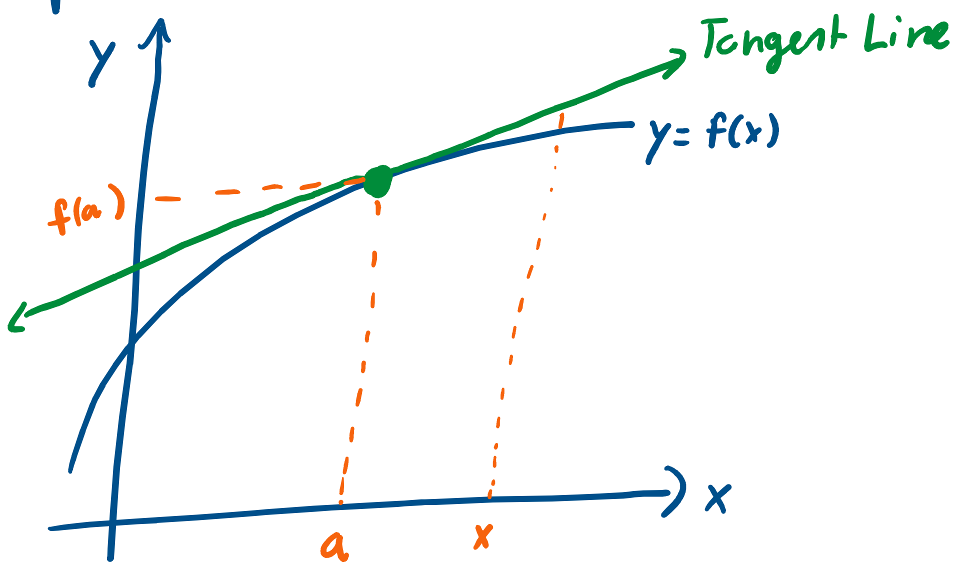
Recall the Slope-Point Formula which states
Given Slope-m and point-(a,b) we have the following equation of a line

$$y-b=m(x-a)$$

Recall the following graph of the function $y=f(x)$ with tangent at the point $(a, f(a))$



If we zoom in to the point $(a, f(a))$ the curve approaches its tangent line at the point. This fact is the key to linear approximation
i.e. We use the tangent line at a point to approximate the value of the function at points near $(a, f(a))$.

Again the tangent line is given by

$$y-f(a)=f'(a)(x-a) \quad \text{for a point } (a, f(a))$$

If we solve for y.

$$y=\underbrace{f(a)+f'(a)(x-a)}_{L(x)\text{ is called the linear approximation}}$$

$y = \underbrace{f(a) + f'(a)(x-a)}$

L(x) is called the linear approximation of f(x) at the x=a.

Again L(x) approximates f(x). So $L(x) \approx f(x)$ when x is CLOSE to a.

Ex 1: Find the linear approximation of $f(x) = \sqrt{x}$ at $x=1$ and use it to approximate $\sqrt{1.01}$.

① First find $f'(x)$.

$$f(x) = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

② Find $f(1)$ and $f'(1)$

$$f(x) = x^{1/2} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(1) = 1 \qquad f'(1) = 1/2$$

③ Plug into the formula w/ $a=1$.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(1) + f'(1)(x-1) \\ &= 1 + \tfrac{1}{2}(x-1) \end{aligned}$$

④ Lastly approximate $\sqrt{1.01}$

i.e. Plug $x=1.01$ into our $L(x)$.

$$\begin{aligned} L(1.01) &= 1 + \tfrac{1}{2}(1.01-1) \\ &= 1 + \tfrac{1}{2}(0.01) \\ &= 1 + \frac{1}{2} \cdot \frac{1}{100} = 1 + \frac{1}{200} = 1.005 \end{aligned}$$

Ex 2: Find the linear approximation to $f(x) = \sin(x)$ at $x = 60°$ and use it to approximate $\sin(59°)$

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① First find $f'(x)$.

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$$

② Find $f(60°)$ and $f'(60°)$

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$

$$f(60°) = \sin(60°) = \sqrt{3}/2 \qquad f'(60°) = \cos(60°) = 1/2$$

③ Plug into the formula w/

Remember $60° = \frac{\pi}{3}$

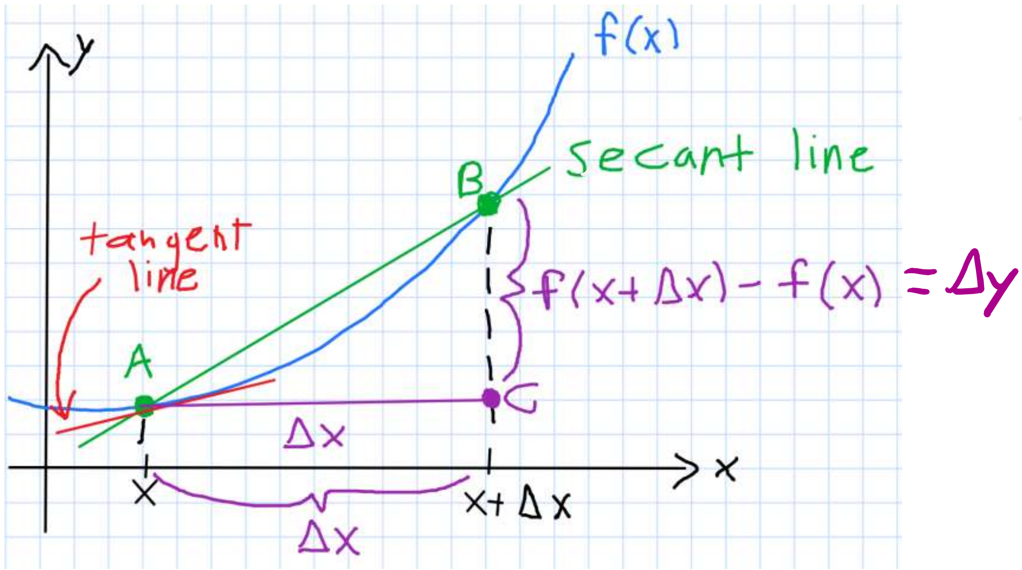
$$L(x) = f(a) + f'(a)(x-a)$$
$$= f(60°) + f'(60°)(x - 60°)$$
$$= \sqrt{3}/2 + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

④ Lastly approximate $\sin(59°)$

i.e. Plug $x = \frac{59}{180}\pi$ into our $L(x)$.

Note: $59° = 59° \cdot \frac{\pi}{180°} = \frac{59}{180}\pi$

$$L\left(\frac{59}{180}\pi\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{59\pi}{180} - \frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{59\pi}{180} - \frac{60\pi}{180}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{-\pi}{180}\right)$$
$$= \frac{\sqrt{3}}{2} - \frac{\pi}{360}$$



Remember

$$\begin{array}{c}\text{Slope of}\\ \text{Tangent Line}\end{array} = \lim_{\Delta x \to 0} \begin{array}{c}\text{Slope of}\\ \text{Secant}\\ \text{line}\end{array} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Another way to see this is

$$\text{Slope} = \frac{\Delta y}{\Delta x} \quad \text{so} \quad \begin{array}{c}\text{Slope of}\\ \text{Tangent Line}\end{array} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Note: dy and dx are known as differentials

So based on these concepts and the picture we say

$\Delta y = f(x+\Delta x) - f(x)$ can be approximated by $dy = f'(x)\,dx$

when $\Delta x = dx$ is close to 0.

Ex 3: Using differentials, calculate the approximate change in $f(x) = 3\cos^2(x)$ given $x = \frac{\pi}{4}$ and $dx = \Delta x = 0.1$.

① Find $x + \Delta x$.

$$x + \Delta x = \frac{\pi}{4} + 0.1$$

② Find $\Delta y = f(x+\Delta x) - f(x)$

$$\Delta y = f\left(\frac{\pi}{4} + 0.1\right) - f\left(\frac{\pi}{4}\right)$$

$$= 3\cos^2\left(\frac{\pi}{4} + 0.1\right) - 3\cos^2\left(\frac{\pi}{4}\right)$$

$\Delta y = f(\frac{\pi}{4} + 0.1) - f(\frac{\pi}{4})$

$$= 3\cos^2\left(\frac{\pi}{4} + 0.1\right) - 3\cos^2\left(\frac{\pi}{4}\right)$$

$$= 3\cos^2\left(\frac{\pi}{4} + 0.1\right) - \frac{3}{2}$$

③ Approximate $\Delta y \approx dy$

$dx \approx \Delta x$

$f(x) = 3\cos^2(x) \iff y = 3[\cos(x)]^2$

$$\frac{dy}{dx} = 3 \cdot 2\cos(x)[-\sin(x)]$$

$$dy = -6\cos(x)\sin(x)\,dx$$

$$dy = -6\cos(x)\sin(x)\,\Delta x$$

So @ $x = \pi/4$ and $x = 0.1$

$$dy = -6\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) \cdot 0.1$$

$$= \frac{-6}{10} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{-6}{10} \cdot \frac{2}{4} = \frac{-12}{40} = \frac{-3}{10}$$

④ Final Step set ② = ③, to get your approximation.

$$\Delta y \approx dy$$

$$3\cos^2\left(\frac{\pi}{4} + 0.1\right) - \frac{3}{2} \approx -\frac{3}{10}$$