Recall the limit

$$\lim_{x\to 4} \frac{x^2-16}{x-4}$$

if I plug in $x=4$ then I get $\lim_{x\to 4} \frac{x^2-16}{x-4} = \frac{0}{0}$.

Note that I could factor and get limit is equal to 8. But what if I can't factor?

"$\frac{0}{0}$" is called indeterminant form.

All indeterminants we will use in this class are

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0},\ \infty-\infty$$

If we have any of these indeterminant forms with some rewriting then we can use the following rule:

<u>L'Hopital Rule:</u> Suppose that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \underline{OR} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. Then we can say

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

<u>Ex 1</u>: Compute the limit of

$$\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$$

$$\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x} = \frac{\sqrt{9}-3}{0} = \text{"}\frac{0}{0}\text{"}$$

Remember,

$$\left((9+3x)^{1/2}\right)' = \frac{1}{2}(9+3x)^{-1/2}(3) = \frac{3}{2\sqrt{9+3x}}$$

By L'Hopital,

$$\lim \sqrt{9+3x}-3 = \lim \frac{3}{2\sqrt{9+3x}} = \frac{3}{2\sqrt{9}} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x} = \lim_{x\to 0} \frac{\frac{3}{2\sqrt{9+3x}}}{1} = \frac{3}{2\sqrt{9}} = \frac{1}{2}$$

Ex 2: Compute the limit

$$\lim_{x\to\infty} \frac{4x^3-6x+1}{2x^3-10x+3}$$

$$\lim_{x\to\infty} \frac{4x^3-6x+1}{2x^3-10x+3} = \text{"}\frac{\infty}{\infty}\text{"}$$

By L'Hopital

$$\lim_{x\to\infty} \frac{4x^3-6x+1}{2x^3-10x+3} = \lim_{x\to\infty} \frac{12x^2-6}{6x^2-10} = \text{"}\frac{\infty}{\infty}\text{"}$$

When this happens again just repeat the process

By L'Hopital

$$\lim_{x\to\infty} \frac{4x^3-6x+1}{2x^3-10x+3} = \lim_{x\to\infty} \frac{12x^2-6}{6x^2-10} = \lim_{x\to\infty} \frac{24x}{12x} = \lim_{x\to\infty} 2 = 2$$

For all the other indeterminant forms you need to do some rewritting of the function to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Ex 3: Compute the limit

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right)$$

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right) = \text{"}\infty \cdot 0\text{"}$$

So let's rewrite the function

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right) = \lim_{x\to\infty} \frac{x^2 \sin\left(\frac{1}{4x^2}\right)}{1} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{\sin\left(1/4x^2\right)}{1/x^2} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\infty} \frac{\sin(1/4x^2)}{1/x^2} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right) = \lim_{x\to\infty} \frac{\sin((4x^2)^{-1})}{x^{-2}}$$

$$= \lim_{x\to\infty} \frac{\cos((4x^2)^{-1})(-1)(4x^2)^{-2}\cdot(8x)}{(-2)x^{-3}}$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{4x^2}\right)\cdot\left(-\frac{8x}{16x^4}\right)}{\frac{-2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{4x^2}\right)\left(+\frac{1}{2}\right)\left(\cancel{\frac{1}{x^3}}\right)}{+2\left(\cancel{\frac{1}{x^3}}\right)}$$

$$= \frac{1}{4}\lim_{x\to\infty} \cos\left(\frac{1}{4x^2}\right) = \frac{1}{4}\cdot\cos(0) = \frac{1}{4}$$

Ex 4: Compute the limits

$$\lim_{x\to\infty} \left(x - \sqrt{x^2-3x}\right)$$

$$\lim_{x\to\infty} \left(x - \sqrt{x^2-3}\right) = \text{“}\infty - \infty\text{”}$$

So let's rewrite the function: Trick here is conjugates

$$\lim_{x\to\infty} \frac{(x-\sqrt{x^2-3x})}{1}\cdot\frac{(x+\sqrt{x^2-3x})}{(x+\sqrt{x^2-3x})} = \lim_{x\to\infty} \frac{x^2-(x^2-3x)}{x+(x^2-3x)^{1/2}}$$

$$= \lim_{x\to\infty} \frac{3x}{x+(x^2-3)^{1/2}} = \text{“}\frac{\infty}{\infty}\text{”}$$

By L'Hopital,

$$\lim_{x\to\infty} \frac{3x}{x+(x^2-3x)^{1/2}} = \lim_{x\to\infty} \frac{3}{1+\frac{1}{2}(x^2-3x)^{-1/2}(2x-3)}$$

$$\lim_{x\to\infty} \frac{3x}{x+(x^2-3x)^{1/2}} = \lim_{x\to\infty} \frac{3}{1+\frac{1}{2}(x^2-3x)^{-1/2}(2x-3)}$$

$$= \lim_{x\to\infty} \frac{3}{1+\frac{2x-3}{2(x^2-3x)^{1/2}}}$$

Use our tricks from limits approaching $\pm\infty$ i.e. Look the leading term since it dominates the limit.

$$= \lim_{x\to\infty} \frac{3}{1+\frac{2x}{2(x^2)^{1/2}}}$$

$$= \lim_{x\to\infty} \frac{3}{1+\frac{2x}{2x}}$$

$$= \lim_{x\to\infty} \frac{3}{1+1} = \frac{3}{2}$$

Ex 5: Compute the limit

$$\lim_{x\to 0^+} x^x$$

$$\lim_{x\to 0^+} x^x = \text{"}0^0\text{"}$$

So let's rewrite the function

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}$$

$$= e^{\text{"}0\cdot(-\infty)\text{"}}$$

Another indeterminant form that's not $0/0$ or ∞/∞.

So keep rewriting

$$e^{\lim_{x\to 0^+} x\ln x} = e^{\lim_{x\to 0^+} \frac{\ln x}{1/x}} = e^{\text{"}\frac{\infty}{\infty}\text{"}}$$

Finally by L'Hopital

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Finally by L'Hopital

$$e^{\lim_{x\to 0^+} \frac{\ln x}{1/x}} = e^{\lim_{x\to 0^+} \frac{1/x}{-1/x^2}} = e^{\lim_{x\to 0^+} \frac{-x^2}{x}} = e^{\lim_{x\to 0^+} -x} = e^{0} = 1$$

Ex 5: Compute the limit

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \text{“}1^\infty\text{”}$$

So let's rewrite the function.

$$\lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to\infty} \frac{x\ln(1+x^{-1})}{1}\cdot\frac{1/x}{1/x}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+x^{-1})}{x^{-1}}} = e^{\text{“}\frac{0}{0}\text{”}}$$

By L'Hopital,

$$e^{\lim_{x\to\infty} \frac{\ln(1+x^{-1})}{x^{-1}}} = e^{\lim_{x\to\infty} \frac{\frac{1}{1+x^{-1}}\cdot\cancel{(-1)x^{-2}}}{\cancel{(-1)x^{-2}}}} = e^{\lim_{x\to\infty} \frac{1}{1+1/x}} = e^{\frac{1}{1+0}} = e$$