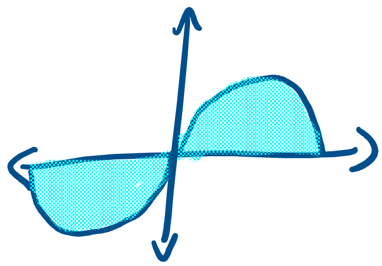
Lesson 28: Approximating the area under curves (Riemann Sums)



Purpose: How to approx the signed area under the curve of a function.

First let's recall some info through some examples.

Ex 1: Evaluate $\sum_{i=1}^{5} i^2 = 1^2+2^2+3^2+4^2+5^2$.

$= 1+4+9+16+25$

$= (1+9)+(4+16)+25$

$= 10+20+25$

$= 55$

Ex 2: Use the Sigma notation to write the sum

$2(1^3+1)+2(2^3+1)+\dots+2(n^3+1)$

Notice the pattern.

All have a multiple of 2.

All have $(\#^3+1)$ and # starts at 1 and ends w/ n.

So $\sum_{i=1}^{n} 2(i^3+1)$

Ex 3: Use the Sigma notation to write the sum

$\frac{2}{0+5}+\frac{2}{1+5}+\dots+\frac{2}{n+5}$

Notice the pattern

All have numerators 2

#+5 and # starts at 0 and ends w/ n.

Notice the pattern:

All have numerators 2

All have denominators #+5 and # starts at 0 and ends w/ n.

So $\sum_{i=0}^{n} \frac{2}{i+5}$

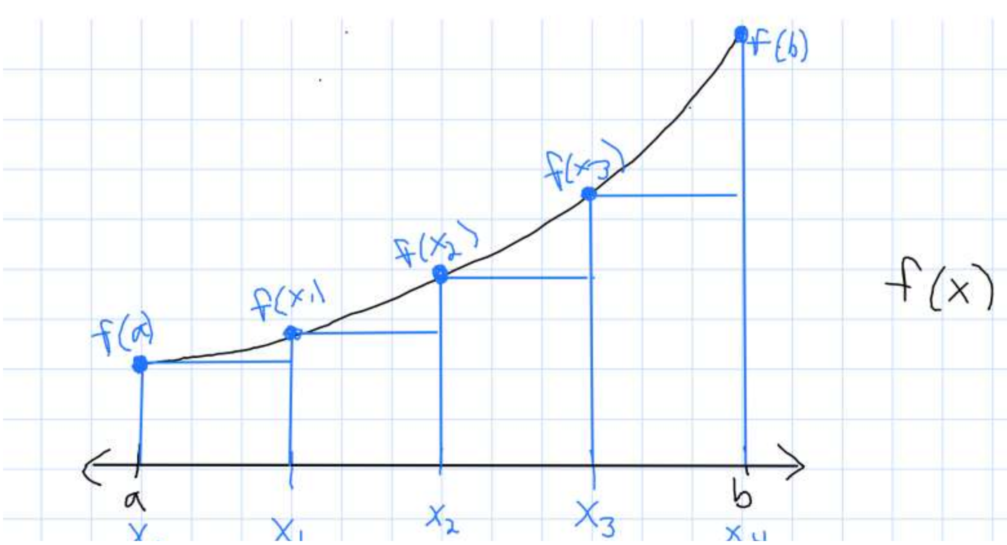
Signed area is the area enclosed by the function and the x-axis with a sign.

- If the function is above the x-axis, then the area is positive.
- If the function is below the x-axis, then the area is negative.

We want to estimate the area under f(x) from a to b by using 4 rectangles.

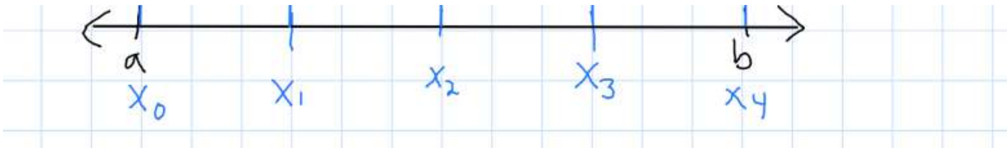
We divide the interval [a,b] into 4 sub-intervals.

Then construct the rectangles by starting at the points on f(x) whose x coordinates are the left end of each of the subintervals.



The total area of these rectangles gives us an estimate of the area under f(x).

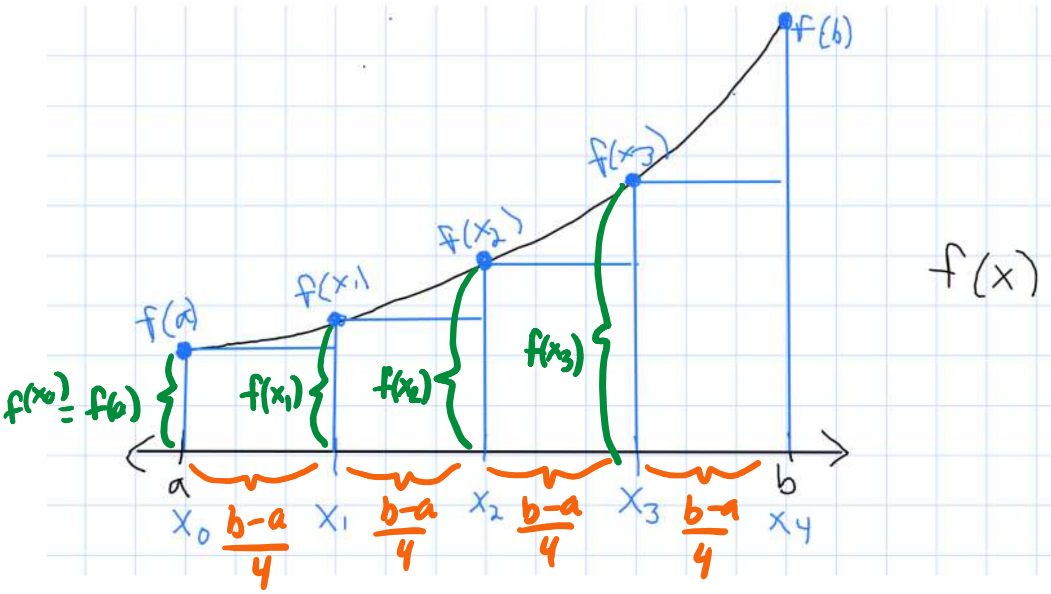
A sum like this is called the Left Riemann Sum.



How do we compute this sum?

Find the area of each rectangle and sum them all up.

BUT let's try to work smart.



Width of each rectangle is the same

$$w = \Delta x = \frac{b-a}{4}$$

Length of first rectangle $f(a) = f(x_0)$

Length of second rectangle $f(x_1)$

$\vdots$

Length of ith rectangle $f(x_i)$

$\Rightarrow$ Area of each rectangle is $lw = \Delta x\, f(x_i)$

So 4 rectangles $\Rightarrow \sum_{i=0}^{3} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$$\Delta x = \frac{b-a}{4}$$

We constructed the rectangles by starting at

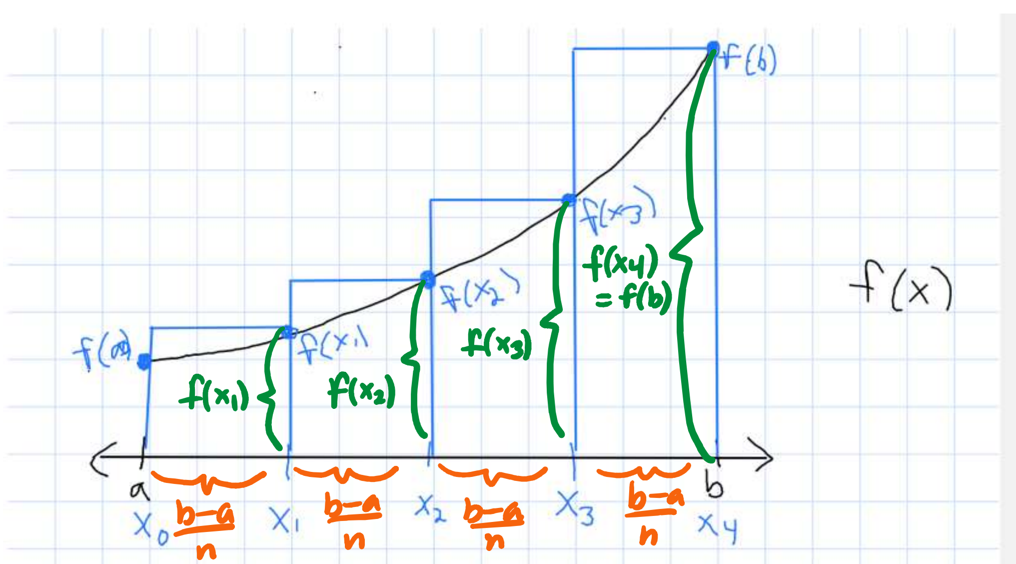
~~Why 3 not 4?~~ We constructed the rectangles by starting at ^{4} the points on $f(x)$ whose x coordinates are the left end of each of the subintervals.

So what is the formula for n rectangles?

Left Riemann Sum $= L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$\Delta x = \dfrac{b-a}{n}$

Can we estimate the area under a curve using the right end of each subinterval? Yes. It's called the Right Riemann Sum.



where $n = 4$

Right Riemann Sum $= R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

where $x_i = a + i\Delta x$

$\Delta x = \dfrac{b-a}{n}$

Note the formulas look the same but aren't the key difference is the index.

The reason is because, as shown in the image we are using the right lengths, shown in green, of each rectangle. So again.

Right Riemann Sum $= R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

$\Delta x = \dfrac{b-a}{n}$

Left ... $L_n = \sum^{n-1} f(x_i)\Delta x$ where $x_i = a + i\Delta x$

Sum

Left Riemann Sum $= L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$ where $x_i = a + i\Delta x$
$\Delta x = \frac{b-a}{n}$

where the formulas are the same except for the indices.

Ex 4: Use the Left Riemann Sum with 3 rectangles to estimate the area under the curve $y = x^2$ on the interval of $[1,7]$.

$a = 1$
$b = 7$
$n = 3$

$\Delta x = \frac{b-a}{n} = \frac{7-1}{3} = \frac{6}{3} = 2$

$x_i = a + i\Delta x = 1 + i \cdot 2 = 1 + 2i$

$f(x_i) = f(1+2i) = (1+2i)^2$

So $L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$L_3 = \sum_{i=0}^{3-1} (1+2i)^2 \cdot 2$

$= 2\sum_{i=0}^{2} (1+2i)^2$

$= 2\left[(1+2(0))^2 + (1+2(1))^2 + (1+2(2))^2\right]$

$= 2\left[1^2 + 3^2 + 5^2\right] = 2(35) = 70$

Ex 5: Use the Right Riemann Sum with 4 rectangles to estimate the area under the curve $y = \sqrt{x} - x$ on the interval of $[0,4]$.

$a = 0$ $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

Note i is a variable

$a=0$ $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{4-0}{4} = 1$

$b=4$

$n=4$ $\quad x_i = a + i\Delta x$

$\qquad = 0 + i \cdot 1 = i$

$\qquad f(x_i) = f(i) = \sqrt{i} - i$

Note i is a variable not imaginary

$$\text{So } R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$R_4 = \sum_{i=1}^{4} \left[\sqrt{i} - i\right] \cdot 1$$

$$= \left[\cancelto{0}{\sqrt{1} - 1} + \sqrt{2} - 2 + \sqrt{3} - 3 + \sqrt{4} - 4\right]$$

$$= \sqrt{2} + \sqrt{3} + \cancelto{0}{2 - 2} - 3 - 4$$

$$= \sqrt{2} + \sqrt{3} - 7$$

Ex 6: Use the Right Riemann Sum with 50 rectangles to estimate the area under the curve $y = e^{5x} + \sin(10x)$ on the interval of $[0, 10]$. Write your answer using sigma notation.

$a=0$ $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{10-0}{50} = \dfrac{10}{50} = \dfrac{1}{5}$

$b=10$

$n=50$ $\quad X_i = a + i\Delta x$

$\qquad = 0 + i(1/5) = i/5$

$\qquad f(x_i) = f\left(\frac{i}{5}\right) = e^{5(i/5)} + \sin\left(10(i/5)\right)$

$\qquad = e^{i} + \sin(2i)$

Remember i is a variable not the imaginary number.

$$\text{So } R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$R_{50} = \sum_{i=1}^{50} \left(e^{i} + \sin(2i)\right) \cdot \frac{1}{5} = \sum_{i=0}^{50} \frac{e^{i} + \sin(2i)}{5}$$

$$R_{50} = \sum_{i=1}^{50} \left(e^{i} + \sin(2i)\right) \cdot \frac{1}{5} = \sum_{i=0} \frac{\cdots}{5}$$