Lesson 29: Definite Integrals

Wednesday, November 5, 2025 11:58 AM

Recall from last time

Left Riemann Sum $= L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

Right Riemann Sum $= R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

where for both sums
$x_i = a + i\Delta x$
$\Delta x = \frac{b-a}{n}$

When the # of rectangles used for the Left/Right Riemann Sums gets bigger and bigger, the approximation of the area under the curve gets better and better.

i.e. The approximation gets closer and closer to the exact signed area.

So what happens when $n \to \infty$ for these sums?

Well the Left/Right Riemann Sum approaches the actual signed area.

i.e. $\lim\limits_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \underbrace{\int_a^b f(x)\,dx}$

Let a be the lower limit of integration
b be the upper limit

Then this is called a definite integral since it has lower and upper limits.

So no more $+C$, when you see lower and upper limits in your integral.

Ex 1: Consider the following limit

$\lim\limits_{\Delta x \to 0} \sum_{i=1}^{n} x_i \ln(x_i)\Delta x$ on $[1,2]$

Express it as a definite integral.

Remember

$\lim \sum^{n} f(x_i)\Delta x = \int^{b} f(x)\,dx$

Remember

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

So $$\lim_{\Delta x \to 0} \sum_{i=1}^{n} x_i \ln(x_i)\Delta x = \int_a^b x\ln(x)\,dx$$

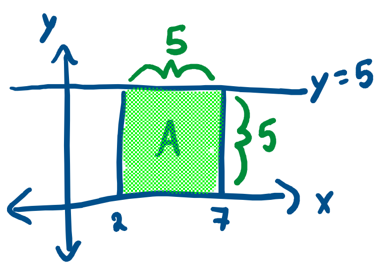
$$= \int_1^2 x\ln(x)\,dx$$ b/c we are in the interval $[1,2]$

So how do we calculate these integrals?

For now using geometry.

Ex 2: Evaluate the definite integral $\int_2^7 5\,dx$ by using geometric formulas.

Note $f(x) = 5$ on $[2,7]$



So $A = 5(5) = 25$

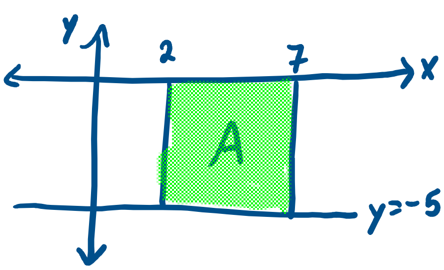
$\int_2^7 5\,dx = 25$

From the graph we see it is a square so find its length and width and use its area formula.

Ex 3: Evaluate the definite integral $\int_2^7 (-5)\,dx$ by using geometric formulas.

Note $f(x) = -5$ on $[2,7]$

(Same comment as in Ex 3)



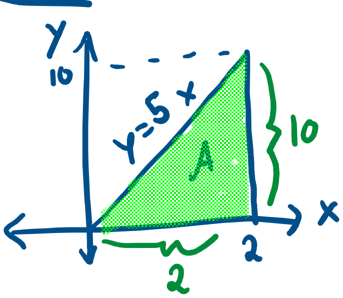
So $A = 5(5) = 25$

But note we are below the x-axis.

So $\int_2^7 (-5)\,dx = -25$

Ex 4: Evaluate the definite integral $\int_0^2 5x\,dx$ by using geometric formulas.

Note $f(x) = 5x$ on $[0,2]$



So $A = \frac{1}{2}(2)(10) = 10$

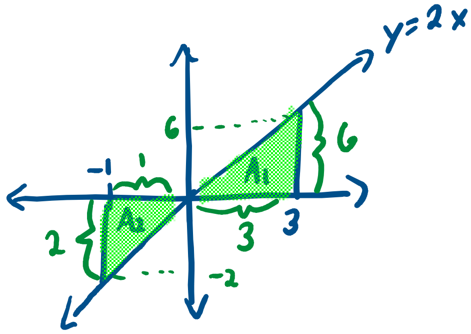
So $\int_0^2 5x\,dx = 10$

From the graph, we see we have a triangle. Find its height and base and plug that in its area formula.

Ex 5: Evaluate the definite integral $\int_{-1}^{3} 2x\,dx$ by using geometric formulas.

Ex 5: Evaluate the definite integral $\int_{-1}^{3} 2x\,dx$ by using geometric formulas.

$\hookrightarrow$ Note $f(x)=2x$ on $[-1,3]$. From the graph, we see we have <u>TWO</u> triangles. So find each height and width to each corresponding triangle, and find each area.



$A_1 = \frac{1}{2}(3)(6) = 9$

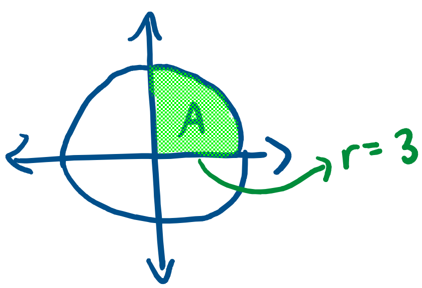
$A_2 = \frac{1}{2}(2)(1) = 1$

Note A_2 is beneath the x-axis, so

$$\int_{-1}^{3} 2x\,dx = A_1 - A_2 = 9 - 1 = 8$$

Ex 6: Evaluate the definite integral $\int_0^3 \underbrace{\sqrt{9-x^2}}\,dx$ using geometric formulas.

$\hookrightarrow$ Note $f(x)=\sqrt{9-x^2}$ on $[0,3]$. From the graph, we have a quarter of a circle.



Area of Full circle: πr^2

$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{\pi(3)^2}{4} \leftarrow \text{quarter piece.}$$

$$= 9\pi/4$$

Properties of Definite Integrals

$\int_a^a f(x)\,dx = 0$

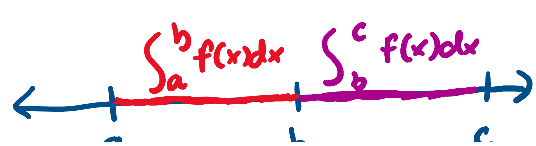
$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

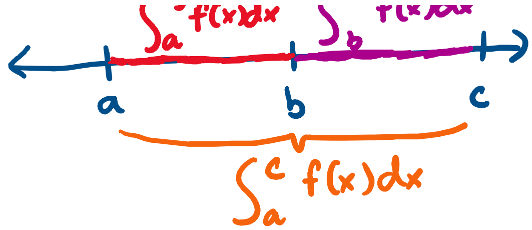
$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ where $b \in [a,c]$

Picture of this Last Property:



Picture of this Last Property:



<u>Ex 7</u>: Given $\int_1^3 f(x)dx = 5$, $\int_3^4 f(x)dx = 2$, $\int_1^3 g(x)dx = 10$. Evaluate the following

ⓐ $\int_1^3 2f(x)dx = 2\underbrace{\int_1^3 f(x)dx}_{=5} = 2(5) = 10$

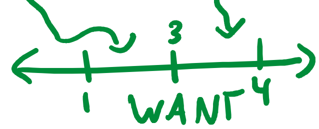
ⓑ $\int_4^3 f(x)dx = -\underbrace{\int_3^4 f(x)dx}_{=2} = -2$

ⓒ $\int_1^3 [2f(x) - 3g(x)]dx = 2\underbrace{\int_1^3 f(x)dx}_{=5} - 3\underbrace{\int_1^3 g(x)dx}_{10}$

$= 2(5) - 3(10) = -20$

ⓓ $\int_1^4 f(x)dx = \underbrace{\int_1^3 f(x)dx}_{=5} + \underbrace{\int_3^4 f(x)dx}_{=2} = 5 + 2 = 7$

Another way to see this is



<u>Ex 8</u>: Given $\int_a^b g(x)dx = 5$ and $\underbrace{\int_a^c g(x)dx = 8\int_a^b g(x)dx}$.

Compute $\int_b^c g(x)dx$.

Recall $\int_a^c g(x)dx = \int_a^b g(x)dx + \int_b^c g(x)dx$

So $\quad 8\int_a^b g(x)dx = \int_a^b g(x)dx + \int_b^c g(x)dx \quad$ b/c

Solve for $\int_b^c g(x)dx$.

$8\int_a^b g(x)dx - \int_a^b g(x)dx = \int_b^c g(x)dx$

$$8\int_a^b g(x)\,dx - \int_a^b g(x)\,dx = \int_b^c g(x)\,dx$$

$$7\int_a^b g(x)\,dx = \int_b^c g(x)\,dx$$

Remember $\int_a^b g(x)\,dx = 5$, so

$$35 = 7(5) = \int_b^c g(x)\,dx$$