Sections 2.1: Idea of Limits
2.2: Definition of Limits

Definition: If $f(x)$ approaches $(\to)$ L as $x \to c$ we say that the limit of $f(x)$ as $x \to c$ is L.
i.e. $\lim_{x \to c} f(x) = L$

Note that f does not need to be defined at $x=c$ for the limit to exist.

Definition: A one-sided limit is the value that the function $f(x) \to L$ as $x \to c$ from the left or right.

- Left-sided Limit: If $f(x) \to L$ as $x \to c$ from the left,
$\lim_{x \to c^-} f(x) = L$
- Right-sided Limit: If $f(x) \to L$ as $x \to c$ from the right
$\lim_{x \to c^+} f(x) = L$

Good news is if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$, then $\lim_{x \to c} f(x) = L$

But how do we find these limits?
We evaluate $f(x)$ at values of x that are getting closer and closer to c and see what happens with the values of the function.

Example 1: Evaluate numerically
ⓐ $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (2x-3)$

ⓑ $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (2x-3)$

ⓒ $\lim_{x \to 4} f(x) = \lim_{x \to 4} (2x-3)$

Plug each # into $f(x)=2x-3$.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	4.8	4.98	4.998	–	5.002	5.02	5.2

ⓐ $\lim\limits_{x\to 4^-} (2x-3) = 5$ b/c the left side is getting closer to 5.

ⓑ $\lim\limits_{x\to 4^+} (2x-3) = 5$ b/c the right side is getting closer to 5.

ⓒ $\lim\limits_{x\to 4} (2x-3) = 5$ b/c ⓐ and ⓑ are the same

But why can't I just plug in 4 into $f(x)=2x-3$?
The reason why is because the function may not be defined at that #.

Example 2: Evaluate numerically

ⓐ $\lim\limits_{x\to 3^-} \dfrac{x^3-3x^2}{x-3}$ ⓑ $\lim\limits_{x\to 3^+} \dfrac{x^3-3x^2}{x-3}$ ⓒ $\lim\limits_{x\to 3} \dfrac{x^3-3x^2}{x-3}$

Plug each # into $f(x) = \dfrac{x^3-3x^2}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	8.41	8.9401	8.994	–	9.006	9.0601	9.6

ⓐ $\lim\limits_{x\to 3^-} \dfrac{x^3-3x^2}{x-3} = 9$ b/c the left side is getting closer to 9.

ⓑ $\lim\limits_{x\to 3^+} \dfrac{x^3-3x^2}{x-3} = 9$ b/c the right side is getting closer to 9.

ⓒ $\lim\limits_{x\to 3} \dfrac{x^3-3x^2}{x-3} = 9$ b/c ⓐ and ⓑ are the same

Finding Limits Graphically

Graphically, we will look at the portion of the curve of f(x) near x=c and see what the function value, y, approaches as x gets closer to c from the left or the right, respectively.

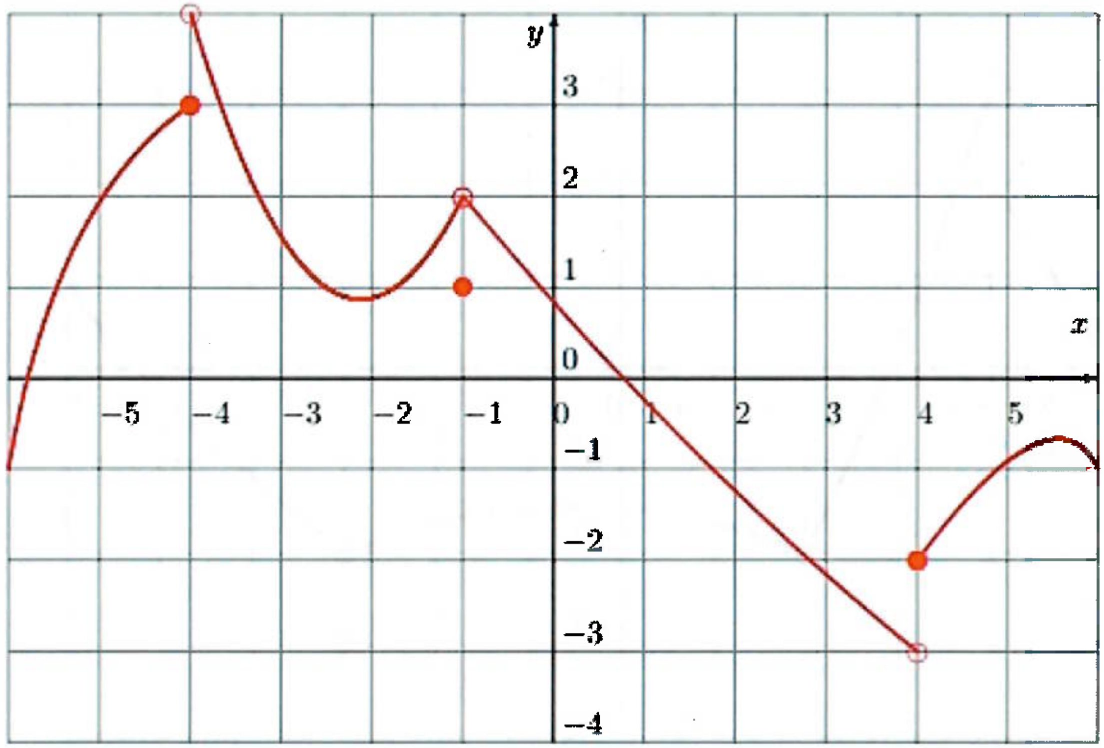
Again if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, then

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c} f(x)} \quad (*)$$

Note that this doesn't imply that $(*) = f(c)$.

Example 1 (From Worksheet)

1. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x\to -4^-} f(x) =$ 3

B) $\lim_{x\to -4^+} f(x) =$ 4

C) $\lim_{x\to -4} f(x) =$ DNE

D) $f(-4) =$ 3

E) $\lim_{x\to -1^-} f(x) =$ 2

F) $\lim_{x\to -1^+} f(x) =$ 2

G) $\lim_{x\to -1} f(x) =$ 2

H) $f(-1) =$ 1

I) $\lim_{x\to 4^-} f(x) =$ −3

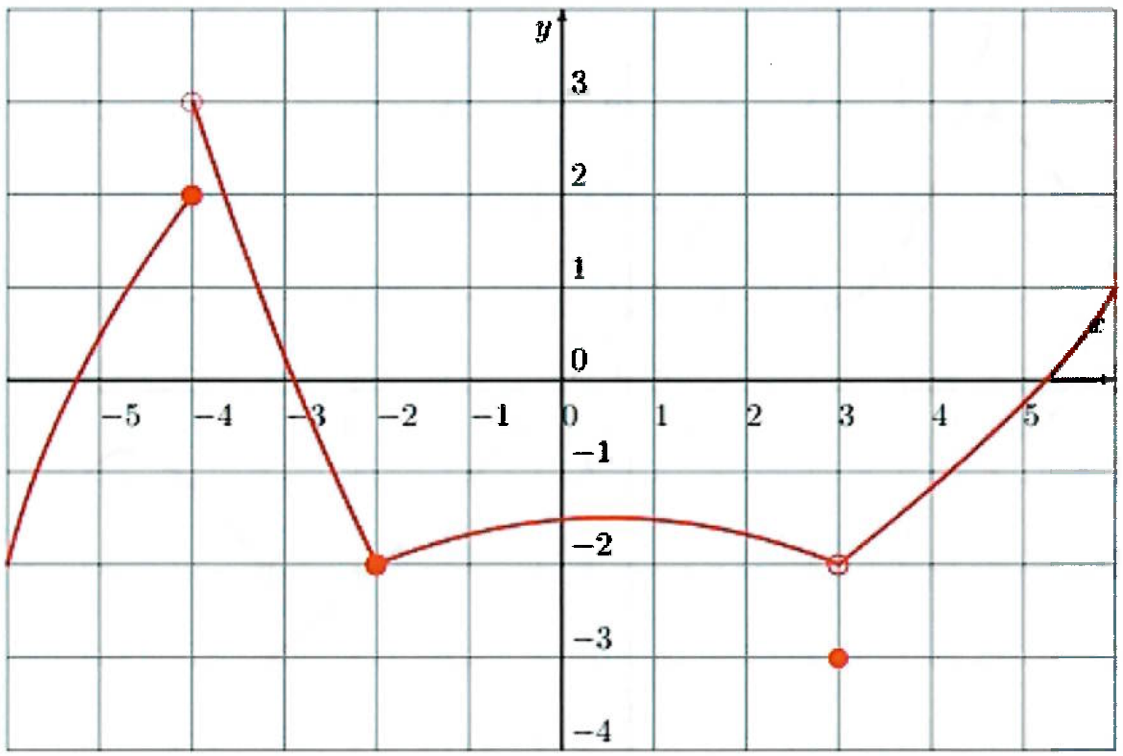
J) $\lim_{x\to 4^+} f(x) =$ −2

K) $\lim_{x\to 4} f(x) =$ DNE

L) $f(4) =$ −2

Example 3 (From Worksheet)

3. Consider the following function defined by its graph:



Find the following limits:

A) $\lim_{x\to -4^-} f(x) = 2$

B) $\lim_{x\to -4^+} f(x) = 3$

C) $\lim_{x\to -4} f(x) =$ DNE

D) $f(-4) = 2$

E) $\lim_{x\to -2^-} f(x) = -2$

F) $\lim_{x\to -2^+} f(x) = -2$

G) $\lim_{x\to -2} f(x) = -2$

H) $f(-2) = -2$

I) $\lim_{x\to 3^-} f(x) = -2$

J) $\lim_{x\to 3^+} f(x) = -2$

K) $\lim_{x\to 3} f(x) = -2$

L) $f(3) = -3$

Section 2.3: Computing the Limits

There are 3 different cases to consider.

① f(c) return a # (it could be 0)

i.e. f(x) is continuous @ $x=c$.

i.e. $\lim_{x\to c} f(x) = f(c)$

Example 1: $\lim_{x\to 4}(2x-3) = 2(4)-3 = 8-3 = 5$

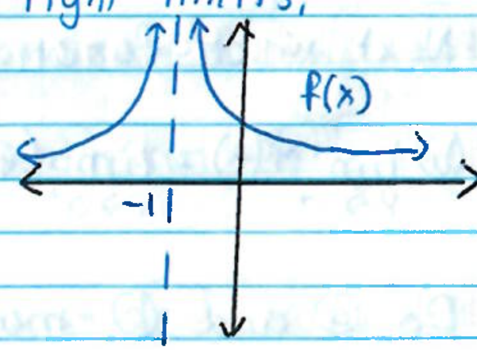
② f(c) returns $\frac{\text{nonzero } \#}{0}$

i.e. f(x) has a Vertical Asymptote @ $x=c$

i.e. $\lim_{x\to c} f(x) = \pm\infty$ or = DNE

Example 2: $\lim_{x\to -1} \frac{1}{(x+1)^2}$

$f(-1) = \frac{1}{(-1+1)^2} = \frac{1}{0} \Rightarrow$ We need to check the left and right limits.



Based on the graph on the right,

$\lim_{x\to -1^-} \frac{1}{(x+1)^2} = \infty$

$\lim_{x\to -1^+} \frac{1}{(x+1)^2} = \infty$

and since both limits match

$\lim_{x\to -1} \frac{1}{(x+1)^2} = \infty$

③ f(c) returns $\frac{0}{0}$

i.e. f(x) has a hole (if a factor cancels out) or

Idea: Manipulate $f(x)$ so that it can look like case 1 or 2.

Example 3: $\lim_{x \to 3} \frac{x^3 - 3x^2}{x - 3}$

$f(3) = \frac{3^3 - 3(3)^2}{3-3} = \frac{27-27}{3-3} = \frac{0}{0} \Rightarrow$ Let's try factoring

$$\lim_{x \to 3} \frac{x^3 - 3x^2}{x - 3} = \lim_{x \to 3} \frac{x^2\cancel{(x-3)}}{\cancel{x-3}} = \lim_{x \to 3} x^2 = 3^2 = 9$$

Example 4: Let $f(x) = \begin{cases} 13x^2 - 6, & x \leq 0 \\ 6x + 6, & x > 0 \end{cases}$

Find the following limits.

ⓐ $\lim_{x \to 0^-} f(x)$ ⓑ $\lim_{x \to 0^+} f(x)$ ⓒ $\lim_{x \to 0} f(x)$

Start by asking yourself which function is the left of 0? $f(x) = 13x^2 - 6$

ⓐ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (13x^2 - 6) = 13(0)^2 - 6 = -6$

Next which function is to the right of 0? $f(x) = 6x + 6$

ⓑ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (6x + 6) = 6(0) + 6 = 6$

Do ⓐ and ⓑ match? No

ⓒ $\lim_{x \to 0} f(x) = \text{DNE}$