Lesson 32: Substitution Rules

Saturday, November 22, 2025 10:28 AM

Warm-Up: Determine the Inner/Outer Functions for the Following

ⓐ $h(x) = \sin^3 x$

Note $h(x) = (\sin x)^3$

So outer function $\Rightarrow f(x) = x^3$

inner function $\Rightarrow g(x) = \sin(x)$

Check $h(x) = f(g(x))$ is true.

$f(g(x)) = f(\sin(x))$

$= (\sin x)^3 = h(x)$ ✓

ⓑ $h(x) = \sqrt{3x+2}$

So outer function $\Rightarrow f(x) = \sqrt{x}$

inner function $\Rightarrow g(x) = 3x+2$

Check $h(x) = f(g(x))$ is true.

$f(g(x)) = f(3x+2)$

$= \sqrt{3x+2} = h(x)$ ✓

ⓒ $h(x) = \tan(3x^2)$

So outer function $\Rightarrow f(x) = \tan(x)$

inner function $\Rightarrow g(x) = 3x^2$

Check $h(x) = f(g(x))$ is true.

$f(g(x)) = f(3x^2)$

$= \tan(3x^2) = h(x)$ ✓

Integration By Substitution is kinda of the Integration Version of the Chain Rule. Also referred to as "Change of Variables."

Ex 1: Find $\int 2x\cos(x^2)\,dx$

Idea to solve Ex 1 is to undo the chain rule.

First determine if you have a function within a function. In this example,

$\cos(x^2)$

where $\cos(x)$ is the outer function, and

x^2 is the inner function

What's the derivative of the inner function? $\underline{2x}$

Do you see $2x$ in the integrand? $\underline{\text{Yes!!!}}$

Let's recall chain rule: $y' = f'(g(x)) \cdot g'(x)$ for $y = f(g(x))$

So from our example: $y' = \cos(x^2) \cdot 2x$

Let's recall chain rule ...

So from our example: $y' = \cos(x^2) \cdot 2x$

i.e. $f(x)$ $\quad g(x) = x^2$

$f'(x) = \cos(x)$ $\quad g'(x) = 2x$

So what's $f(x)$?

$$f(x) = \int f'(x)\,dx = \int \cos(x)\,dx = \sin(x)$$

Hence $y = f(g(x)) = \sin(x^2)$

i.e. $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

How to fast track this method?

Do a change of variable using the inner function.

So let's redo Ex 1 the quicker way

$$\int \cos(\underbrace{x^2}_{u}) \cdot \underbrace{2x\,dx}_{du} \;\frac{u = x^2}{du = 2x\,dx}\; \int \cos(u)\,du = \sin(u) + C$$

But if we start with a function of x. We want to end with a function of x. So sub back $u = x^2$.

$$\int \cos(x^2) \cdot 2x\,dx = \sin(x^2) + C$$

Note we may not see what du equals in our integrand. So you may have to do some equation manipulation.

<u>Ex 2:</u> Compute the following integrals:

(a) $\int \sqrt{4x+1}\,dx$

$$\int \sqrt{4x+1}\,dx \;\frac{u = 4x+1}{du = 4\,dx}\; \int \sqrt{u}\,\frac{du}{4} = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} = \frac{1}{4} \cdot \frac{2}{3}(4x+1)^{3/2} + C$$

$$\int \sqrt{4x+1}\,dx \quad \frac{}{du = 4\,dx} \quad \int \sqrt{u}\cdot\frac{du}{4} = \frac{1}{4}\int u \ldots$$

$$\frac{du}{4} = dx$$

$$= \frac{1}{6}(4x+1)^{3/2} + C$$

ⓑ $\int 3x^5 e^{x^6}\,dx$

$$\int 3x^5 e^{x^6}\,dx \quad \frac{u = x^6}{du = 6x^5\,dx} \quad \int 3\cancel{x^5} e^u \cdot \frac{du}{6\cancel{x^5}} = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{x^6} + C$$

$$\frac{du}{6x^5} = dx$$

ⓒ $\int \frac{3\sin(x)}{\cos^8(x)}\,dx$

$$\int \frac{3\sin(x)}{(\cos x)^8}\,dx = \int 3\sin(x)[\cos(x)]^{-8}\,dx \quad \frac{u = \cos(x)}{du = -\sin(x)\,dx} \quad \int 3\cancel{\sin(x)}\,u^{-8}\cdot\left(\frac{du}{-\cancel{\sin(x)}}\right)$$

$$\frac{du}{-\sin(x)} = dx$$

$$= -3\int u^{-8}\,du = -3\,\frac{u^{-8+1}}{-8+1} = -3\,\frac{(\cos(x))^{-7}}{-7} + C = \frac{3}{7\cos^7(x)} + C$$

Sometimes we need to do extra work with u, as shown in the next example(s).

Ex 3: Compute the following integrals:

ⓐ $\int x\sqrt{x-5}\,dx$

$$\int x\sqrt{x-5}\,dx \quad \frac{u = x-5}{du = dx} \quad \int x\sqrt{u}\,du$$

Issue: Our integral has x and u! Can I rewrite x into some function of u? Yes!

$u = x-5 \iff u+5 = x$

$$\underline{\underline{x = u+5}} \quad \int (u+5)u^{1/2}\,du$$

$$= \int u^{3/2} + 5u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} + 5\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}u^{5/2} + 5\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}(x-5)^{5/2} + \frac{10}{3}(x-5)^{3/2} + C$$

ⓑ $\displaystyle\int \frac{2x}{(x+7)^{3/2}}\,dx$

$$\int 2x(x+7)^{-3/2}\,dx \;\frac{u=x+7}{du=dx}\; \int 2x\,(u)^{-3/2}du$$

Note: $u=x+7$ is the same as $u-7=x$

$$\underline{\underline{x=u-7}}\;\int 2(u-7)\,u^{-3/2}du$$

$$= \int 2u^{-1/2} - 14u^{-3/2}\,du$$

$$= 2\cdot\frac{2}{1}u^{1/2} - 14\cdot\frac{2}{-1}u^{-1/2}$$

$$= 4(x+7)^{1/2} + 28(x+7)^{-1/2} + C$$

Now lets do integration by substitution for definite integrals.

Ex 3: Compute

$$\int_0^2 \sqrt{4x+1}\,dx$$

Previously we found $\int \sqrt{4x+1}\,dx = \frac{1}{6}(4x+1)^{3/2} + C$

Issue: What do I do with the bounds? (i.e. $\int_0^2$)

Well $\displaystyle\int_0^2 \sqrt{4x+1}\,dx = \int_{x=0}^{x=2} \sqrt{4x+1}\,dx$

Method 1: Change $x=0$ and $x=2$ using $u=4x+1$

$$x=0 \downarrow u=1 \qquad x=2 \downarrow x=9$$

So $\displaystyle\int^{x=2} \sqrt{4x+1}\,dx = \int^{u=9} \frac{1}{4}u^{1/2}du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2}\Big]^{u=9}$

So $\int_{x=0}^{x=2} \sqrt{4x+1}\,dx = \int_{u=1}^{u=9} \frac{1}{4} u^{1/2}\,du = \frac{1}{4}\cdot\frac{2}{3} u^{3/2}\Big]_{u=1}^{u=9}$

$= \frac{1}{6}\left[9^{3/2} - 1^{3/2}\right] = \frac{1}{6}\left[27-1\right] = \frac{26}{6} = \frac{13}{3}$

Method 2: To treat the integral after the u-sub as indefinite and when you plug u back evaluate with original bounds.

$\int_{x=0}^{x=2} \sqrt{4x+1}\,dx \;\frac{u=4x+1}{du=4dx}\; \int \frac{1}{4} u^{1/2}\,du = \frac{1}{4}\cdot\frac{2}{3} u^{3/2} = \frac{1}{6}(4x+1)^{3/2}\Big]_{x=0}^{x=2}$

$= \frac{1}{6}\left((4(2)+1)^{3/2} - (4(0)+1)^{3/2}\right) = \frac{13}{3}$