Lesson 33: Exponential Models (Growth & Decay)

Saturday, November 22, 2025 11:22 AM

Let's recall how to solve Initial Value Problem (IVP) through the following example.

Ex1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Try getting all y terms to one side.

$$\frac{dy}{y} = 2dt$$

Integrate both sides

$$\int \frac{dy}{y} = \int 2dt$$

$$\ln|y| = 2t + C$$

Solve for y.

$$e^{\ln|y|} = e^{2t+C}$$

$$|y| = e^{2t}e^{C}$$

$$\pm y = e^{C}e^{2t}$$

$$y = \pm e^{C} e^{2t}$$

all of this is a constant

$$y = Ce^{2t}$$

Now let's use the initial condition $y(0) = 100$ to get C.

$$100 = Ce^{2\cdot 0}$$

$$100 = Ce^{0}$$

$$100 = C$$

So $y = 100e^{2t}$

Exponential Models

If y is a differential function of t such that

$$\frac{dy}{dt} = y' = ky \quad \text{for some constant } k$$

Then $y = Ce^{kt}$ where C is a constant.

Note: $e^{x} = \exp[x]$

Note: k is proportionality constant or growth rate or decay rate

C is the initial value of y

If $k > 0$ and $c > 0$, then this model is called the exponential growth model

If $k < 0$ and $c > 0$, then this model is called the exponential decay model.

Ex 2: In a saving account where the interest is compounded continuously, if the initial investment is \$500 and the annual interest rate is 3% how much money will there be in 10 years?

much money will there be in 10 years?

Note $C = 500 > 0$ and $k = 0.03 > 0$

So $y = 500 \exp[0.03t]$

$y(10) = 500 \exp[0.03(10)] \approx \$674.93$

How long does it take to double the initial investment?

Previously, we found $y = 500\exp[0.03t]$

Double the initial investment is $2(500)$

So $$2(\cancel{500}) = \cancel{500}\exp[0.03t]$$

$$2 = \exp[0.03t]$$

$$\ln 2 = 0.03t$$

$$\frac{\ln 2}{0.03} = t$$

$$t \approx 23.1 \text{ years}$$

Ex 3: In a savings account where the interest is compounded continuously, if the initial investment is \$100 and there are \$150 in 8 years, what is the annual interest rate?

Goal of the problem: Solving for k.

k - interest rate > 0 and $C = 100 > 0 \Rightarrow y = 100e^{kt}$

The question also states $y(8) = 150$.

$$150 = 100e^{k(8)}$$

$$\frac{150}{100} = e^{8k}$$

$$\frac{3}{2} = e^{8k}$$

$$\ln\left(\frac{3}{2}\right) = 8k$$

$$\frac{1}{8}\ln\left(\frac{3}{2}\right) = k$$

$$k \approx 0.05 \Rightarrow 5\%$$

Remember interest rate is always a percent.

Ex 5: The population of a country follows exponential model and the continuous

Ex 5: The population of a country follows exponential model and the continuous annual rate of change k of the population is -0.001. The population is 10 million when $t=2$. What happens when $t=6$?

$$k=-0.001<0 \text{ and } C>0 \Rightarrow y=C\exp[-0.001t]$$

We are also told $y(2)=10$.

$$10=C\exp[-0.001(2)]$$

$$10=C\exp[-0.002]$$

$$10\exp[0.002]=C\exp[-0.002]\exp[0.002]$$

$$10\exp[0.002]=C\cancel{\exp[\underbrace{-0.002+0.002}_{0}]} \to 1$$

$$C=10\exp[0.002]$$

So

$$y=10\exp[0.002]\exp[-0.001t]$$

$$y=10\exp[0.002-0.001t]$$

So when $t=6$

$$y(6)\approx 9.9601 \text{ millions}$$

Radioactive Isotopes & Half life

Radioactive isotopes decay over time. Hence it follows the exponential decay model. For each of these isotopes the decay rate is unique and characterized by their half-life.

i.e. Half-life of a radioactive isotope is the time that it takes for the isotope to reduce to half of its original quantity.

Again the decay model for these isotopes is

$$y=C\exp[kt]$$

By definition of half-life

$$\frac{1}{2}\cancel{C}=\cancel{C}\exp[k(\text{half-life})]$$

$$\ln\left(\tfrac{1}{2}\right)=k(\text{half-life})$$

$$\frac{\ln(\frac{1}{2})}{\text{half-life}}=k$$

(Note sometimes this is written as $\boxed{\dfrac{-\ln(2)}{\text{half-life}}}=k$)

$$\frac{\ln(\frac{1}{2})}{\text{half-life}} = k$$

(Note sometimes this is written as $\boxed{\frac{-\ln(2)}{\text{half-life}}} = k$)

Note < 0 which makes sense since its a decay rate

Ex 6: The radioactive isotope ^{226}Ra has a half-life of 1,599 years. If there are 10 grams of ^{226}Ra initially, how much is there after 1,000 years?

Like we said the model is $y = C\exp[kt]$ with $k = \frac{-\ln(2)}{\text{half-life}}$

So in this case $k = -\frac{\ln(2)}{1599}$ and $C = 10$

So our eqn is $y = 10\exp\left[-\frac{\ln(2)}{1599}t\right]$

So 1000 years later, $y(1000) \approx 6.4828$ grams