Lesson 8: Derivative as a Function

Monday, September 15, 2025 10:25 AM

So far, we have used the derivative to find the slope of the tangent line at a single point. But what if we want to know the slope for every x? This leads us to the derivative function.

<u>Def.</u> The derivative of a function $f(x)$ is itself a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Ex.</u> Let $f(x) = -x^2 + 6x$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(-(x+h)^2 + 6(x+h)) - (-x^2 + 6x)}{h}$$

$$= \lim_{h \to 0} \frac{(-(x^2 + 2xh + h^2) + 6(x+h)) - (-x^2 + 6x)}{h}$$

$$= \lim \cancel{-x^2} - 2xh - h^2 + \cancel{6x} + 6h + \cancel{x^2} - \cancel{6x}$$

$$= \lim_{h \to 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{6x} + 6h + \cancel{x^2} - \cancel{6x}}{h}$$

$$= \lim_{h \to 0} \frac{-2xh - h^2 + 6h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x + h + 6)}{\cancel{h}}$$

$$= \lim_{h \to 0} (-2x + h + 6)$$

$$= -2x + 6$$

So $f'(x) = -2x + 6$ which in itself is a function.

Domain of the Derivative

Ex: Let $f(x) = \sqrt{x}$, which has domain $[0, \infty)$. Compute $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\quad \lim_{h\to 0} \frac{\quad}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h} \cdot \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{(x+h)-\cancel{\sqrt{x}\sqrt{x+h}}+\cancel{\sqrt{x}\sqrt{x+h}}-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

So $f'(x) = \frac{1}{2\sqrt{x}}$ but note that $f'(x)$ is defined only from $(0,\infty)$. So just because the domain of $f(x)$ is $[0,\infty)$ doesn't mean $f'(x)$ is also defined there.

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Graphical Interpretation of f'

we can understand the derivative function $f'(x)$ by looking at the graph of $f(x)$

- Where $f(x)$ is increasing, $f'(x) > 0$.
- Where $f(x)$ is decreasing, $f'(x) < 0$.
- Where $f(x)$ has a horizontal tangent, $f'(x) = 0$.

Ex: Using $f(x) = \begin{cases} -3x-5, & x \le -2 \\ x+3, & -2 < x \le 0 \\ -\frac{1}{2}x+3, & x > 0 \end{cases}$

determine how the graph of f' looks.

$\underline{x \le -2}$: $f(x) = -3x-5$

and has a negative slope which means its decreasing. So $f'(x) < 0$ when $x \le -2$.

$\underline{-2<x\leq 0}$: $f(x)=x+3$

and has a positive slope which means its increasing

So $f'(x)>0$ when $-2<x\leq 0$

$\underline{x>0}$: $f(x)=-\frac{1}{2}x+3$

and has a negative slope which means its decreasing.

So $f'(x)<0$ when $x>0$.

Continuity vs Differentiability

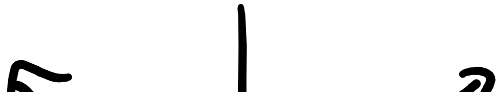
__Def:__ A function f is differentiable at $x=a$ if $f'(a)$ exists.

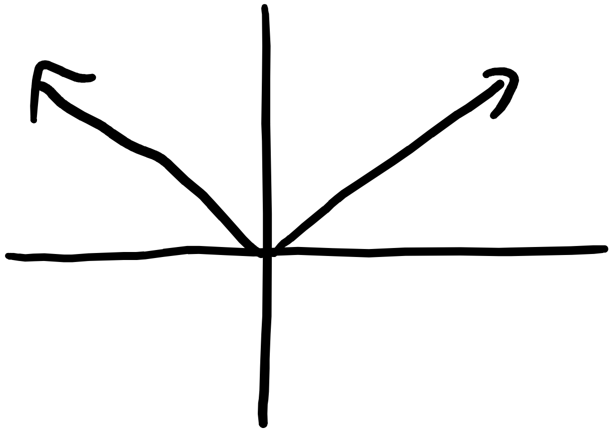
__Note:__ Differentiability $\Rightarrow$ Continuity

__BUT__ Continuity $\not\Rightarrow$ Differentiability always.

__Ex:__ Let $f(x)=|x|$. The graph is

continuous everywhere cause my pen never





cause my pen never lifted from the paper

<u>BUT</u> at $x=0$ it has a sharp corner

When there is a sharp corner $\Rightarrow$ not differentiable. So the derivative doesn't exist there.

<u>Key takeaway:</u> Differentiability requires both smoothness and continuity.