

MATH 165

STUDY GUIDE EXAM 3

Please show all your work! Answers without supporting work will not be given credit.
Write answers in spaces provided.

Solutions

Name: _____

1. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$ and the values on the critical value(s) $f(c)$. You should know what the definition of a critical value is.

Example Problems

- (1.1) Find the absolute maximum/minimum and local maximum/minimum of the function defined by $f(x) = 3x^4 - 16x^3 + 18x^2$ on the closed interval $[-1, 4]$.

$$\begin{aligned} f'(x) &= 12x^3 - 48x^2 + 36x = 0 \\ 12x[x^2 - 4x + 3] &= 0 \\ 12x(x-1)(x-3) &= 0 \\ x &= 0, 1, 3 \end{aligned}$$

Abs max/min

x	f(x)	Conclusion
-1	$3+16+18=37$	Max
1	$3-16+18=5$	
0	0	
3	-27	Min
4	32	

Rel max/min: $f' = (12x)(x-1)(x-3)$

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$$\begin{aligned} f'(-1) &= (-)(-)(-) \\ f'(1/2) &= (+)(-)(-) \\ f'(2) &= (+)(+)(-) \\ f'(3) &= (+)(+)(+) \end{aligned}$$

$$\begin{aligned} &3 \cdot 4^4 - 16 \cdot 4^3 + 18 \cdot 4^2 \\ &4^2[3 \cdot 16 - 16 \cdot 4 + 18] \\ &4^2[-16 + 18] = 32 \end{aligned}$$

Absolute Max at: $x = -1$ Absolute Min at: $x = -3$ Local Max at: $x = 1$ Local Min at: $x = 0, 3$

(1.2) Find the absolute maximum and absolute minimum values of the function f on the given interval.

(1.2.1) $f(x) = x^{2/3}$ on $[-1, 8]$

$$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3} \cdot \frac{1}{x^{1/3}} = 0$$

but this never happens

x	$f(x)$	Conclusion
-1	$(-1)^{2/3} = 1^{1/3} = 1$	Min
8	$8^{2/3} = 2^2 = 4$	Max

Absolute Max at: $x = 8$

Absolute Min at: $x = -1$

(1.2.2) $f(x) = xe^{-x}$ on $[-4, 6]$

$$\begin{aligned} f'(x) &= e^{-x} + xe^{-x}(-1) \\ &= e^{-x}(1-x) = 0 \\ &\quad \underbrace{\text{Never}}_{=0} \quad x=0 \end{aligned}$$

x	$f(x)$	Conclusion
-4	$e^4(-3) \approx -163$	Min
1	0	Max
6	$e^{-6}(-5) \approx -0.01$	

Absolute Max at: $x = 1$

Absolute Min at: $x = -4$

(1.2.3) $f(x) = x^2 e^{-x}$ on $[-4, 6]$

$$f'(x) = 2x e^{-x} + x^2 e^{-x} (-1)$$

$$= x \underbrace{e^{-x}}_{\text{Never } = 0} (2-x) = 0$$

So $x=0, 2$

x	$f(x)$	Conclusion
-4	$16e^4 \approx 873$	Max
0	0	Min
2	$4e^{-2} \approx 0.5$	
6	$36e^{-6} \approx 0.3$	

Absolute Max at: $x = -4$

Absolute Min at: $x = 0$

(1.2.4) $f(x) = (x^2 - 1)^3$ on $[-1, 3]$

$$f'(x) = 3(x^2 - 1)(2x) = 0$$

$$x = \pm 1, 0$$

x	$f(x)$	Conclusion
-1	0	
0	-1	Min
1	0	
3	8^3	Max

Absolute Max at: $x = 3$

Absolute Min at: $x = 0$

(1.2.5) $f(x) = 2 \cos(x) + \sin(2x)$ on $[0, \pi/2]$

$$\begin{aligned} f'(x) &= -2 \sin(x) + \cos(2x) \cdot 2 = 0 \\ -2 \sin(x) + (1 - 2 \sin^2(x)) \cdot 2 &= 0 \\ \sin(x) + 1 - 2 \sin^2(x) &= 0 \\ 0 &= 2 \sin^2(x) - \sin(x) - 1 \\ &= (2 \sin(x) + 1)(\sin(x) - 1) \end{aligned}$$

$\pi/2$ $\star$
S | A

x	f(x)	Conclusion
0	2	Max
$\pi/2$	0	Min

$$\begin{aligned} 2 \sin(x) + 1 &= 0 \\ \sin(x) &= -1/2 \\ \text{Never happens} & \text{ in } [0, \pi/2] \end{aligned}$$

$$\begin{aligned} \sin(x) - 1 &= 0 \\ \sin(x) &= 1 \\ x &= \pi/2 \end{aligned}$$

Absolute Max at: $x = 0$

Absolute Min at: $x = \pi/2$

(1.2.6) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$

$$\begin{aligned} f'(x) &= \frac{2x+1}{x^2+x+1} = 0 \\ 2x+1 &= 0 \\ x &= -1/2 \end{aligned}$$

$$\frac{1}{4} - \frac{1}{2} + 1 = \frac{1}{4} + \frac{1}{2} =$$

x	f(x)	Conclusion
-1	$\ln(1) = 0$	
$-1/2$	$\ln(3/4) \approx -0.28$	Min
1	$\ln(3) \approx 1.09$	Max

Absolute Max at: $x = 1$

Absolute Min at: $x = -1/2$

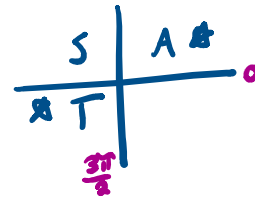
(1.2.7) $f(x) = e^{-x} \cdot \sin(x)$ on $[0, 3\pi/2]$

$$f'(x) = -e^{-x} \sin(x) + e^{-x} \cos(x) = 0$$

$$e^{-x} [\sin(x) + \cos(x)] = 0$$

$$\cos(x) = \sin(x)$$

$$x = \frac{\pi}{4}, \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$



x	$f(x)$	Conclusion
0	0	Min
$\pi/4$	$e^{-\pi/4}(\sqrt{2}/2) \approx 0.1$	
$5\pi/4$	$e^{-5\pi/4}(\sqrt{2}/2)$	
$3\pi/2$	$e \approx 2.1$	Max

Absolute Max at: $x = 3\pi/2$

Absolute Min at: $x = 0$

(1.3) Let $f(x)$ be the function given by

$$f(x) = 4 \sin^3(x) + 21 \sin^2(x) - 24 \sin(x)$$

Find its absolute maximum value and absolute minimum value on the interval. $[-\pi/2, \pi/2]$.

$$f'(x) = 12 \sin^2(x) \cos(x) + 42 \sin(x) \cos(x) - 24 \cos(x) = 0$$

$$6 \cos(x) [2 \sin^2(x) + 7 \sin(x) - 4] = 0$$

$$6 \cos(x) [2 \sin(x) - 1] [\sin(x) + 4]$$



$$\begin{array}{lll} \cos x = 0 & \sin(x) = 1/2 & \sin(x) = -4 \\ x = -\pi/2, \pi/2 & x = \pi/6 & \text{Never} \end{array}$$

x	$f(x)$
$-\pi/2$	$-4 + 21 + 24 = 41$
$\pi/6$	$4 \cdot \frac{1}{8} + 21 \cdot \frac{1}{4} - 24 \cdot \frac{1}{2} = 2 + 10.5 - 12 = 0.5$
$\pi/2$	$4 + 21 - 24 = 1$

Absolute Max at: $x = -\pi/2$

Absolute Min at: $x = \pi/6$

2. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

Example Problems:

- (2.1) The first derivative of a function f is given by

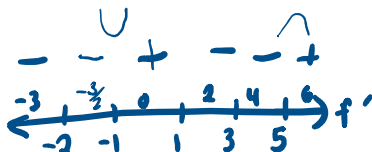
$$f'(x) = (x+2)^2(x+1)(x-1)^3(x+3)^2(x-5) = 0$$

Find the values of x for which the function f takes

$$\Rightarrow x = -2, -1, 1, 3, 5$$

(a) Local maximum and

(b) Local minimum



Local Max at: $x = 5$

Local Min at: $x = -1$

- (2.2) (a) Find the critical numbers of the function $f(x) = x^8(x-4)^7$?

$$f'(x) = 8x^7(x-4)^7 + x^8(7)(x-4)^6 = 0$$

$$x^7(x-4)^6[8x + 7(x-4)] = 0$$

$$x^7(x-4)^6(15x-28) = 0$$

Critical number(s): $x = 0, 4, 4/3$

- (b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?

$$f''(x) = 7x^6(x-4)^6(15x-28) + x^7(6)(x-4)^5(15x-28) + x^7(x-4)^6 \cdot 15$$

$$f''(4/3) = 0 + 0 + (4/3)^7(4/3-4)^6 \cdot 15 > 0$$

Answer: $x = 4/3$ is a rel min

- (c) What does the First Derivative Test tell you that the Second Derivative test does not?

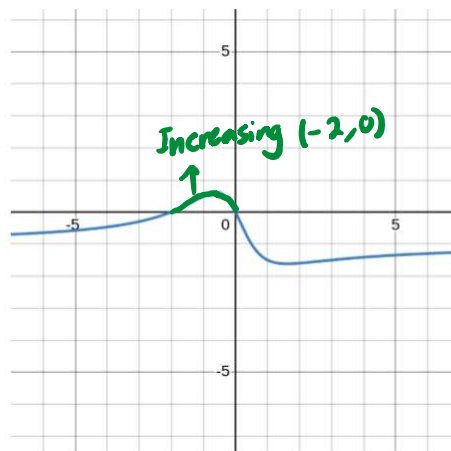


Answer: $x = 0$ is a rel max
 $x = 4$ is a rel min

3. You are supposed to be able to draw the (rough) graph of a function, given the graph of its derivative.

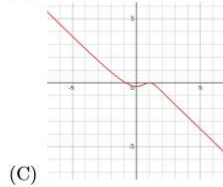
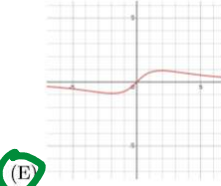
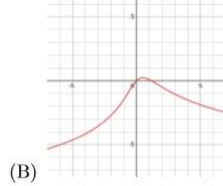
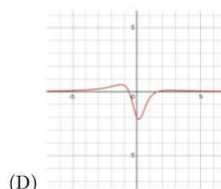
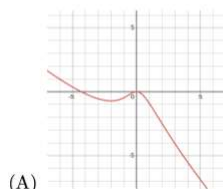
Example Problems:

(3.1) Below is given the graph of $y = f'(x)$.



Decreasing $(-2, 0)$
 $\cup (0, 2)$

Which of the following can be the graph of $y = f(x)$?



4. You are supposed to tell whether the graph of a function is concave up/down, and find its inflection points, by looking at the second derivative of the function.

Example Problems:

- (4.1) Determine how the concavity changes for the function

$$f(x) = \frac{1}{2}x - \sin(x)$$

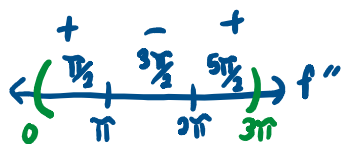
on the interval $(0, 3\pi)$.

$$f'(x) = \frac{1}{2} - \cos(x)$$

$$f''(x) = \sin(x) = 0$$

$$x = \pi, 2\pi$$

h/c $(0, 3\pi)$



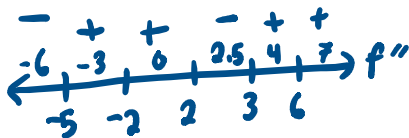
Concavity changes at $x = \pi, 2\pi$

- (4.2) The second derivative of a function f is given by

$$f''(x) = (x+5)^3(x-2)^2(x-3)^3(x+6)^2 = 0$$

Find the x -coordinates of all the inflection points.

$$\Rightarrow x = -5, -2, 2, 3, 6$$



Remember inflection pts happen when there is a concavity change.

$$x = -5, 2, 3$$

(4.3) How many inflection points does the graph of the function $y = f(x) = x^5 - 5x^4 + 25x$ have?

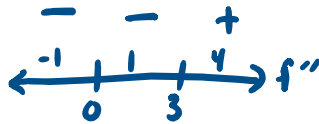
$$f'(x) = 5x^4 - 20x^3 + 25$$

$$f''(x) = 20x^3 - 60x^2 = 0$$

$$20x^2(x-3) = 0$$

$$x = 0, 3$$

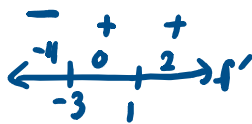
Remember inflection pts happen when concavity change.



Number of inflection points: 1

(4.4) We have a function whose first derivative is given by the formula $f'(x) = (x-1)^2(x+3)^3$. Find the local extrema and the inflection points of the function.

Rel max/
min



$$= 0$$

$$x = 1, -3$$

Inflection pts

$$f''(x) = 2(x-1)(x+3)^3 + (x-1)^2(3)(x+3)^2$$

$$= (x-1)(x+3)^2 [2(x+3) + 3(x-1)]$$

$$= (x-1)(x+3)^2 (5x+3) = 0$$

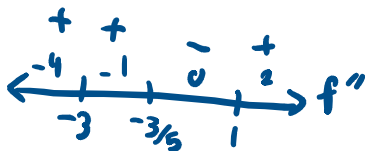
$$x = 1, -3, -3/5$$

- + +

Local Max: N/A

Local Min: $x = -3$

Inflection Points: $x = -3/5, 1$

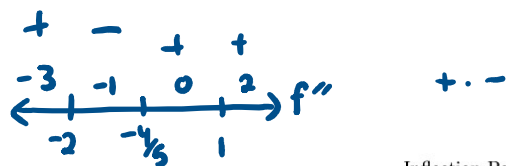


(4.5) We have a function whose first derivative is given by the formula

$$f'(x) = (x-1)^3(x+2)^2$$

Find the value(s) of the x -coordinate of the inflection points of the graph of the function $y = f(x)$.

$$\begin{aligned} f''(x) &= 3(x-1)^2(x+2)^2 + (x-1)^3(2)(x+2) = 0 \\ (x-1)^2(x+2)[3(x+2) + 2(x-1)] &= 0 \\ (x-1)^2(x+2)(5x+4) &= 0 \\ x &= 1, -2, -4/5 \end{aligned}$$



Inflection Points: $x = -2, -4/5$

5. You are supposed to know how to compute the limits using L'Hopital's Rule, under the provision that the limits are formally of the form $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.

Example Problems:

(5.1) Compute the following limits.

$$(5.1.1) \lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$\begin{aligned} &\text{By L'Hopital} \\ \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0 \end{aligned}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}} = 0$$

$$(5.1.2) \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{3x^2} = \frac{0}{0}$$

By L'Hopital

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{6x} = \frac{1}{6} \lim_{x \rightarrow 0} \underbrace{\frac{\sin x}{x}}_{=1} = \frac{1}{6}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{3x^2} = \underline{\underline{1/6}}$$

$$(5.1.3) \lim_{x \rightarrow 0} \frac{e^{7x} - \cos(2x)}{\tan(3x)} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{7e^{7x} + \sin(2x) \cdot 2}{3\sec^2(3x)} = \frac{7+0}{3 \cdot 1} = \frac{7}{3}$$

$$\lim_{x \rightarrow 0} \frac{e^{7x} - \cos(2x)}{\tan(3x)} = \underline{\underline{7/3}}$$

$$(5.1.4) \lim_{x \rightarrow 0} \frac{\sin(x)}{1-x^2} = \frac{0}{1} = 0$$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{1-x^2} = \underline{0}$$

$$(5.1.5) \lim_{x \rightarrow 0} \frac{3x - \sin(3x)}{5x - \tan(5x)} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{3 - 3\cos(3x)}{5 - 5\sec^2(5x)} = \frac{3-3}{5-5} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{+3 \cdot 3 \sin(3x)}{-5 \cdot 2 \sec(5x) \sec(5x) \tan(5x) \cdot 5} = \frac{0}{-5 \cdot 2 \cdot 1 \cdot 1 \cdot 0 \cdot 5} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{9 \sin(3x)}{-50 \sec^2(5x) \tan(5x)}$$

By L'Hopital

$$\lim_{x \rightarrow 0} \frac{27 \cos(3x)}{-100 \sec(5x) \sec(5x) \tan(5x) \tan(5x) \cdot 5 - 50 \sec^2(5x) \sec^2(5x) \cdot 5} = \frac{-\frac{27}{250}}{-50 \cdot 5}$$

$$(5.1.6) \lim_{x \rightarrow 0} \frac{2x - \tan(2x)}{4x - \sin(4x)} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{2 - \sec^2(2x) \cdot 2}{4 - \cos(4x) \cdot 4} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{-2 \cdot 2 \sec(2x) \sec(2x) \tan(2x) \cdot 2}{4 \cdot 4 \sin(4x)} = \frac{0}{0}$$

Rewrite

$$\lim_{x \rightarrow 0} \frac{-2 \sec^2(2x) \tan(2x)}{4 \sin(4x)}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{-4 \sec(2x) \sec(2x) \tan(2x) \tan(2x) \cdot 2 - 2 \sec^2(2x) \sec^2(2x) \cdot 2}{16 \cos(4x)} = \frac{\lim_{x \rightarrow 0} \frac{2x - \tan(2x)}{4x - \sin(4x)}}{\frac{-4}{16}} = \frac{-4}{16} = -\frac{1}{4}$$

$$(5.1.7) \lim_{x \rightarrow 0} \frac{\tan(x) - x}{x^3} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{\sec^2(x) - 1}{3x^2} = \frac{0}{0}$$

By L'Hopital

$$\lim_{x \rightarrow 0} \frac{2 \sec(x) \tan(x) \sec(x)}{6x} = \frac{0}{0}$$

Rewrite

$$\lim_{x \rightarrow 0} \frac{\sec^2(x) \tan(x)}{3x}$$

By L'Hopital

$$\lim_{x \rightarrow 0} \frac{2 \sec(x) \sec(x) \tan(x) + \sec^2(x) \sec^2(x)}{3} \lim_{x \rightarrow 0} \frac{\tan(x) - x}{x^3} = \frac{1}{3}$$

$$= \frac{1}{3}$$

$$(5.1.8) \lim_{x \rightarrow 0} \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2} = \frac{\ln(1)}{0} = \frac{0}{0}$$

b/c $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$

Rewrite

$$\lim_{x \rightarrow 0} \frac{\ln(\sin(x)) - \ln(x)}{x^2}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{\frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{x}}{2x} = \lim_{x \rightarrow 0} \frac{x \cos(x) - \sin(x)}{2x^2 \sin(x)} = \frac{0}{0}$$

By L'Hopital

$$\lim_{x \rightarrow 0} \frac{\cancel{\cos(x)} - x \sin(x) - \cancel{\cos(x)}}{4x \sin(x) + 2x^2 \cos(x)} = \lim_{x \rightarrow 0} \frac{-\sin(x)}{4 \sin(x) + 2x \cos(x)} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{-\cos(x)}{4 \cos(x) + 2 \cos(x) - 2x \sin(x)} = \frac{-1}{6} \lim_{x \rightarrow 0} \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2} = \frac{-1}{6}$$

(5.1.9) $\lim_{x \rightarrow 0} \frac{\ln(\cos(5x))}{x^2} = \frac{0}{0}$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{\frac{1}{\cos(5x)} \cdot -\sin(5x)(5)}{2x} = \lim_{x \rightarrow 0} \frac{-5 \tan(5x)}{2x} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{-5 \sec^2(5x) \cdot 5}{2} = \frac{-25}{2}$$

$$\lim_{x \rightarrow 0} \frac{\ln(\cos(5x))}{x^2} = \frac{-25}{2}$$

$$(5.1.10) \quad \lim_{x \rightarrow \infty} \frac{\overbrace{\tan^{-1}(x) - \frac{\pi}{2}}^{\pi/2}}{\frac{1}{x}} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} -\frac{x^2}{1+x^2} = -\frac{\infty}{\infty}$$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{-2x}{2x} = -1$$

$$\lim_{x \rightarrow \infty} \frac{\tan^{-1}(x) - \frac{\pi}{2}}{\frac{1}{x}} = \underline{-1}$$

6. You are suppose to know how to compute the limits of the form $\pm\infty \times 0, \infty - \infty$

Example Problems:

(6.1) Compute the following limits.

$$(6.1.1) \quad \lim_{x \rightarrow 0} x \cdot \ln(2x) = 0 \cdot \infty$$

Rewrite: $\lim_{x \rightarrow 0} \frac{\ln(2x)}{1/x} = \frac{\infty}{\infty}$

By L'Hopital,

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0} -\frac{x^2}{x} = \lim_{x \rightarrow 0} -x = 0$$

$$\lim_{x \rightarrow 0} x \cdot \ln(2x) = \underline{0}$$

$$(6.1.2) \lim_{x \rightarrow 0^+} x \cdot \ln\left(3 + \frac{5}{x}\right) = 0 \cdot \infty$$

Rewrite: $\lim_{x \rightarrow 0^+} \frac{\ln(3 + \frac{5}{x})}{1/x} = \frac{\infty}{\infty}$

By L'Hopital,

$$\lim_{x \rightarrow 0^+} \frac{\frac{1}{3 + \frac{5}{x}} \cdot \frac{-5}{x^2}}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{5}{3 + \frac{5}{x}} = \lim_{x \rightarrow 0^+} \frac{5x}{3x + 5} = \frac{0}{5} = 0$$

$$\lim_{x \rightarrow 0^+} x \cdot \ln\left(3 + \frac{5}{x}\right) = \underline{0}$$

$$(6.1.3) \lim_{x \rightarrow \infty} 2x \cdot \tan\left(\frac{1}{3x}\right) = 0 \cdot \infty$$

Rewrite: $\lim_{x \rightarrow \infty} \frac{\tan(1/3x)}{1/2x} = \frac{0}{0}$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{\sec^2(1/3x) \cdot \frac{1}{3} \cdot \frac{-1}{x^2}}{\frac{1}{2} \cdot \frac{-1}{x^2}} = \frac{2}{3} \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{3x}\right) = \frac{2}{3}$$

$$\lim_{x \rightarrow \infty} 2x \cdot \tan\left(\frac{1}{3x}\right) = \underline{2/3}$$

$$(6.1.4) \quad \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (2x - \pi) \cdot \tan(x) = 0 \cdot \infty$$

Rewrite: $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\tan(x)}{\frac{1}{2x - \pi}} = \frac{\infty}{\infty}$

By L'Hopital,

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\sec^2(x)}{\frac{-2}{(2x - \pi)^2}} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} - \frac{\sec^2(x) (2x - \pi)^2}{2} = \frac{0}{2} = 0$$

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (2x - \pi) \cdot \tan(x) = \underline{0}$$

$$(6.1.5) \quad \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (\pi - 2x) \cdot \tan(x) = 0 \cdot \infty$$

Rewrite

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\tan(x)}{\frac{1}{\pi - 2x}} = \frac{\infty}{\infty}$$

By L'Hopital,

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\sec^2(x)}{\frac{-2}{(\pi - 2x)^2}} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{(\pi - 2x)^2 \sec^2(x)}{2} = \frac{0}{2}$$

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (\pi - 2x) \cdot \tan(x) = \underline{0}$$

$$(6.1.6) \lim_{x \rightarrow \infty} (\sqrt{x^2 - 5x + 7} - x) = \infty - \infty$$

Rewrite

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 5x + 7} - x}{1} \cdot \frac{\sqrt{x^2 - 5x + 7} + x}{\sqrt{x^2 - 5x + 7} + x} = \lim_{x \rightarrow \infty} \frac{x^2 - 5x + 7 - x^2}{\sqrt{x^2 - 5x + 7} + x}$$

By L'Hopital, $\frac{-\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{2}(x^2 - 5x + 7)^{-1/2}(2x - 5)}{2\sqrt{x^2 - 5x + 7}} = \lim_{x \rightarrow \infty} \frac{-5}{2\sqrt{x^2 - 5x + 7}}$$

$$= \lim_{x \rightarrow \infty} \frac{10(x^2 - 5x + 7)^{1/2}}{2x - 5} = \frac{\infty}{\infty}$$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{10(\frac{1}{2})(x^2 - 5x + 7)^{-1/2}(2x - 5)}{2} = \frac{5}{2} \lim_{x \rightarrow \infty} \frac{2x - 5}{\sqrt{x^2 - 5x + 7}} = \frac{5}{2} \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2 - 5x + 7}}$$

$$= \frac{5}{2} \cdot 2 = 5$$

$$(6.1.7) \lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) = \infty - \infty$$

$$\lim_{x \rightarrow 1} \frac{x \ln(x) - (x-1)}{(x-1) \ln(x)} = \lim_{x \rightarrow 1} \frac{x \ln(x) - x + 1}{(x-1) \ln(x)} = \frac{0}{0}$$

By L'Hopital

$$\lim_{x \rightarrow 1} \frac{\ln(x) - x \cdot \frac{1}{x} - 1}{\ln(x) + (x-1) \cdot \frac{1}{x}} = \lim_{x \rightarrow 1} \frac{\ln(x) - 2}{\ln(x) + \frac{x-1}{x}} = \frac{0 - 2}{0 + \frac{0}{1}} = \frac{-2}{0} = -\infty$$

$$\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) = -\infty$$

$$(6.1.8) \lim_{x \rightarrow 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4} \right) = \infty - \infty$$

$$\lim_{x \rightarrow 4} \left(\frac{\sqrt{x}+2}{(\sqrt{x}+2)(\sqrt{x}-2)} - \frac{4}{x-4} \right) = \lim_{x \rightarrow 4} \frac{\sqrt{x}+2-4}{x-4} = \lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4} = \frac{0}{0}$$

By L'Hopital,

$$\lim_{x \rightarrow 4} \frac{\frac{1}{2}x^{-1/2}}{1} = \lim_{x \rightarrow 4} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

$$\lim_{x \rightarrow 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4} \right) = \underline{\underline{1/4}}$$

$$(6.1.9) \lim_{x \rightarrow \infty} x^2 \cdot \tan\left(\frac{1}{5x^2+2}\right) = \infty \cdot 0$$

$$\lim_{x \rightarrow \infty} \frac{\tan\left(\frac{1}{5x^2+2}\right)}{1/x^2} = \frac{0}{0}$$

By L'Hopital

$$\lim_{x \rightarrow \infty} \frac{\sec^2\left(\frac{1}{5x^2+2}\right) \cdot \frac{-1}{(5x^2+2)^2} (10x)}{\frac{-2}{x^3}} = \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{5x^2+2}\right) \cdot \lim_{x \rightarrow \infty} \frac{10x^4}{(5x^2+2)^2}$$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{40x^3}{2(5x^2+2)(10x)} = \lim_{x \rightarrow \infty} \frac{2x^2}{5x^2+2} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} x^2 \cdot \tan\left(\frac{1}{5x^2+2}\right) = \underline{\underline{2/5}}$$

By L'Hopital,

$$\lim_{x \rightarrow \infty} \frac{4x}{10x} = \lim_{x \rightarrow \infty} \frac{4}{10} = \frac{2}{5}$$

$$(6.1.10) \lim_{x \rightarrow 0} x^2 \cdot \tan\left(\frac{1}{5x^2 + 2}\right) = 0 \cdot \tan\left(\frac{1}{2}\right) = 0$$

$$\lim_{x \rightarrow 0} x^2 \cdot \tan\left(\frac{1}{5x^2 + 2}\right) = \underline{0}$$

7. You are supposed to be able to compute the limits $\lim_{x \rightarrow 0} [f(x)]^{g(x)}$ of the form $0^0, \infty^0, 1^\infty$.

Example Problems:

(7.1) Compute the following limits:

$$(7.1.1) \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{7x} = 1^\infty$$

$$\exp\left[\lim_{x \rightarrow \infty} 7x \ln\left(1 + \frac{3}{x}\right)\right] = \exp\left[\lim_{x \rightarrow \infty} \frac{\ln(1 + 3x^{-1})}{1/(7x)}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \infty} \frac{\frac{1}{1+3x^{-1}}(-3x^{-2})}{-\frac{1}{(7x)^2} \cdot 7}\right] = \exp\left[\lim_{x \rightarrow \infty} \frac{+3}{\left(1 + \frac{3}{x}\right) \cdot x^2} \cdot \frac{(7x)^2}{+7}\right]$$

$$= \exp\left[\lim_{x \rightarrow \infty} \frac{3}{7} \cdot \frac{49x^2}{x^2 + 3x}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \infty} 21 \cdot \frac{2x}{2x+1}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital

$$\exp\left[\lim_{x \rightarrow \infty} \frac{42}{2}\right] = e^{21}$$

$$\ln(x+3) - \ln(x-2)$$

$$(x+3)^{7x+5} = (\infty)^\infty$$

||

$$(7.1.2) \lim_{x \rightarrow \infty} \left(\frac{x+3}{x-2} \right)^{7x+5} = \left(\frac{\infty}{\infty} \right)^{\infty}$$

$$\exp \left[\lim_{x \rightarrow \infty} (7x+5) \ln \left(\frac{x+3}{x-2} \right) \right] = \exp \left[\lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x+3}{x-2} \right)}{\frac{1}{7x+5}} \right] = \frac{\infty}{0}$$

By L'Hopital,

$$\exp \left[\lim_{x \rightarrow \infty} \frac{\frac{1}{x+3} - \frac{1}{x-2}}{\frac{-1}{(7x+5)^2} \cdot 7} \right] = \exp \left[\lim_{x \rightarrow \infty} \frac{x-2-x-3}{(x+3)(x-2)} \cdot \frac{(7x+5)^2}{-7} \right]$$

$$= \exp \left[\lim_{x \rightarrow \infty} \frac{+5(7x+5)^2}{+7(x^2+x-6)} \right] = \frac{\infty}{\infty}$$

By L'Hopital,

$$\exp \left[\lim_{x \rightarrow \infty} \frac{5(7x+5) \cdot 2 \cdot 7}{7(2x+1)} \right] = \frac{\infty}{\infty}$$

$$(7.1.3) \lim_{x \rightarrow 0^+} (1-5x)^{1/x} = 1^{\infty}$$

$$\exp \left[\lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1-5x) \right] = \exp \left[\lim_{x \rightarrow 0^+} \frac{\ln(1-5x)}{x} \right] = \exp \left[\frac{0}{0} \right]$$

By L'Hopital,

$$\exp \left[\lim_{x \rightarrow 0^+} \frac{\frac{1}{1-5x} \cdot (-5)}{1} \right] = \exp \left[\lim_{x \rightarrow 0^+} \frac{-5}{1-5x} \right] = \exp \left[\frac{-5}{1} \right]$$

$$\text{By L'Hopital, } \exp \left[\lim_{x \rightarrow \infty} \frac{5 \cdot 7 \cdot 2}{2} \right] = e^{35}$$

$$\lim_{x \rightarrow \infty} \left(\frac{x+3}{x-2} \right)^{7x+5} = e^{35}$$

$$\lim_{x \rightarrow 0^+} (1-5x)^{1/x} = e^{-5}$$

$$(7.1.4) \lim_{x \rightarrow \infty} (e^x + x)^{1/x} = \infty^0$$

$$\exp\left[\lim_{x \rightarrow \infty} \frac{1}{x} \ln(e^x + x)\right] = \exp\left[\lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{x}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \infty} \frac{\frac{1}{e^x + x} (e^x + 1)}{1}\right] = \exp\left[\lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital

$$\exp\left[\lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \infty} \frac{e^x}{e^x}\right] = \exp\left[\lim_{x \rightarrow \infty} 1\right] = e^1$$

$$\lim_{x \rightarrow \infty} (e^x + x)^{1/x} = \underline{\quad e \quad}$$

$$(7.1.5) \lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{2x}} = \infty^0$$

$$\exp\left[\lim_{x \rightarrow \infty} \frac{1}{2x} \ln(e^x + x)\right] = \exp\left[\lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{2x}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \infty} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{2}\right] = \exp\left[\frac{1}{2} \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x}\right] = \exp\left[\frac{1}{2} \cdot \frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\frac{1}{2} \cdot \lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1}\right] = \exp\left[\frac{1}{2} \cdot \frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\frac{1}{2} \cdot \lim_{x \rightarrow \infty} \frac{e^x}{e^x}\right] = \exp\left[\frac{1}{2} \lim_{x \rightarrow \infty} 1\right] = e^{\frac{1}{2}}$$

$$\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{2x}} = \underline{\quad e^{\frac{1}{2}} \quad}$$

$$(7.1.6) \quad \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (5 \tan(x))^{\cos(x)} = \infty^0$$

$$\exp\left[\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \cos(x) \ln[5 \tan(x)]\right] = \exp\left[\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\ln(5 \tan(x))}{\sec(x)}\right] = \exp\left[\frac{\infty}{\infty}\right]$$

By L'Hopital,

$$\exp\left[\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\frac{1}{5 \tan(x)} \cdot 5 \sec^2(x)}{\sec(x) \tan(x)}\right] = \exp\left[\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\cos(x)}}{\frac{\sin(x)}{\cos(x)}}\right]$$

$$= \exp\left[\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \frac{\cos(x)}{(\sin(x))^2}\right] = \exp\left[\frac{0}{1}\right] = 1$$

$$\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (5 \tan(x))^{\cos(x)} = \underline{1}$$

$$(7.1.7) \quad \lim_{x \rightarrow 0^+} (e^{6/x} - 8x)^{x/2} = \infty^0$$

$$\exp\left[\lim_{x \rightarrow 0^+} \frac{x}{2} \ln(e^{6/x} - 8x)\right] = \exp\left[\lim_{x \rightarrow 0^+} \frac{x}{2} \ln(e^{6/x} (1 - 8xe^{-6/x}))\right]$$

$$= \exp\left[\lim_{x \rightarrow 0^+} \frac{x}{2} (\ln e^{6/x} + \ln(1 - 8xe^{-6/x}))\right]$$

$$= \exp\left[\lim_{x \rightarrow 0^+} \frac{x}{2} \cdot \frac{6}{x} + \frac{x}{2} \ln(1 - 8xe^{-6/x})\right]$$

$$= \exp[3 + 0 \cdot \ln(1 - 0)] = e^3$$

$$\lim_{x \rightarrow 0^+} (e^{6/x} - 8x)^{x/2} = \underline{e^3}$$

8. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply the following corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise:

If $f'(x) = 0$ for all values of $x \in (a, b)$, then a continuous function f on the closed interval $[a, b]$ is actually a constant.

Example Problems:

- (8.1) Consider the function $f(x) = x^4 - 2x^2 + 7x - 2$ over the interval $[-2, 2]$. Does it satisfy the conditions for the Mean Value Theorem to hold? If it does, find the value(s) $c \in (-2, 2)$ such that

$$f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$$

$$f'(x) = 4x^3 - 4x + 7$$

$$\begin{aligned} f(2) - f(-2) &= (2)^4 - 2(2)^2 + 7(2) - 2 - ((-2)^4 - 2(-2)^2 + 7(-2) - 2) \\ &= 16 - 8 + 14 - 2 - (16 - 8 - 14 - 2) = 28 \end{aligned}$$

$$4x^3 - 4x + 7 = \frac{28}{4} = 7$$

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$4x(x-1)(x+1) = 0 \Rightarrow x = 0, -1, 1 \text{ since all in } (-2, 2)$$

Answer: $x = 0, -1, 1$

- (8.2) Consider the function $y = f(x) = x^{2/3}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem hold? Do we have any value $c \in (-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$?

f is continuous on $[-1, 1]$

$$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3 \cdot \sqrt[3]{x}}$$

Note $f'(x)$ doesn't exist at $x=0$. So not differentiable over $(-1, 1)$.

$\Rightarrow$ MVT doesn't hold

Continuous & differentiable everywhere
 $\Downarrow$
MVT holds

(8.3) Determine the exact value of $\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)$.

Note $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Let

$$f(x) = \sin^{-1}(x) + \cos^{-1}(x)$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

B/c $f'(x) = 0$ then by IVT, we know $f(x)$ is constant.

So choose a point in $[-1, 1]$

to evaluate $f(x)$, and that constant w/ be the value over all values in $(-1, 1)$.

(8.4) Determine the exact value of $\sin^{-1}\left(\frac{3}{7}\right) - \cos^{-1}\left(-\frac{3}{7}\right)$.

Note $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Let

$$f(x) = \sin^{-1}(x) - \cos^{-1}(-x)$$

Note $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$$f(x) = \sin^{-1}(x) - \pi + \cos^{-1}(x)$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

B/c $f'(x) = 0$ then by IVT, we know $f(x)$ is constant.

So choose a pt in $[-1, 1]$ to

evaluate $f(x)$, and that constant w/ be the value in $(-1, 1)$.

Let $x = 0$.

$$f(0) = \sin^{-1}(0) + \cos^{-1}(0)$$

$$= 0 + \pi/2$$

$$= \pi/2$$

$$f(1/5) = \pi/2$$

$$\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right) = \pi/2$$

Let $x = 0$.

$$f(0) = \sin^{-1}(0) - \pi + \cos^{-1}(0)$$

$$= 0 - \pi + \pi/2$$

$$= -\frac{\pi}{2}$$

$$f(3/7) = -\pi/2$$

$$\sin^{-1}\left(\frac{3}{7}\right) - \cos^{-1}\left(-\frac{3}{7}\right) = -\pi/2$$

(8.5) Determine the exact value of $\tan^{-1}(7) - \tan^{-1}\left(-\frac{1}{7}\right)$.

Note: $-\tan^{-1}(-a) = \tan^{-1}(a)$

So $\tan^{-1}(7) + \tan^{-1}\left(\frac{1}{7}\right)$

Note: $\tan^{-1}(a) + \tan^{-1}(b) = \frac{\pi}{2}$ if $a \cdot b = 1$

$$7 \cdot \frac{1}{7} = 1$$

So $\tan^{-1}(7) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{2}$

$$\tan^{-1}(7) - \tan^{-1}\left(-\frac{1}{7}\right) = \underline{\pi/2}$$

(8.6) Determine the exact value of $\tan^{-1}(-5) - \tan^{-1}\left(\frac{1}{5}\right)$.

Note: $\tan^{-1}(-a) = -\tan^{-1}(a)$

$$-\tan^{-1}(5) - \tan^{-1}\left(\frac{1}{5}\right)$$

$$= -\left[\tan^{-1}(5) + \tan^{-1}\left(\frac{1}{5}\right)\right]$$

Note: $\tan^{-1}(a) + \tan^{-1}(b) = \frac{\pi}{2}$ if $a \cdot b = 1$

$$5 \cdot \frac{1}{5} = 1$$

So $-\left[\tan^{-1}(5) + \tan^{-1}\left(\frac{1}{5}\right)\right]$
 $= -\pi/2$

$$\tan^{-1}(-5) - \tan^{-1}\left(\frac{1}{5}\right) = \underline{-\pi/2}$$

-
- (8.7) Consider the equation $f(x) = x^3 + x - 1 = 0$. Determine how many solutions are there for the above equation on the interval $[0, 1]$, using the Intermediate Value Theorem and the Mean Value Theorem.

MVT: $f'(x) = 3x^2 + 1$

$$f(1) - f(0) = 1^3 + 1 - 1 - (0^3 + 0 - 1) = 2$$

$$3x^2 + 1 = \frac{2}{1-0}$$

$$3x^2 + 1 = 2$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm \sqrt{\frac{1}{3}}$$

Note only $x = \sqrt{\frac{1}{3}} \in (0, 1)$

Hence there is only one solution.

Answer: _____

(8.8) Consider the function $y = f(x)$ where $f(x)$ is a cubic polynomial (i.e., a polynomial of degree 3 of the form $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$). We know $f(5) = 0$ and $f'(x) = x^2 + 1$. Choose the correct statement from below about the number of solutions (i.e., roots) for the equation $f(x) = 0$.

- (A) Since a cubic polynomial should have 3 distinct roots, we should have two more roots other than 5.
- (B) A cubic polynomial should have 3 roots counted with multiplicity. That is to say, it can have 1 triple root, one double root and 1 simple root, or 3 simple roots. So from the above information alone, it is possible to have one more or even two more roots other than 5.
- (C) We cannot conclude anything about the number of roots for $f(x) = 0$ only from looking only at the 1st derivative $f'(x)$.
- (D) 5 is the ONLY root. Suppose there is another root $r \neq 5$ with $f(r) = 0$. Then the Mean Value Theorem would imply $f'(s) = \frac{f(s) - f(5)}{s - 5} = 0$ for some value s between r and 5. On the other hand, $f'(s) = s^2 + 1 \geq 1 > 0$, a contradiction!
- (E) Since $f'(x) = x^2 + 1 > 0$, this means that the function f is always increasing. Therefore, we should have one more root $r < 5$ with $f(r) = 0$ other than 5.

$$f'(x) = x^2 + 1 > 0 \text{ for every } x$$

$\Rightarrow f$ is strictly increasing everywhere

$\Rightarrow$ The function can cross the x-axis at most once.

Since $f(5) = 0$ that crossing is at $x = 5$, there are no other real roots.

9. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down), and also by determining the horizontal, vertical asymptotes and x-intercept (y-intercept).

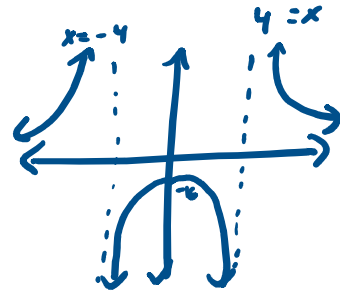
Example Problems:

(9.1) Draw the graph of the following function:

$$(9.1.1) \ y = f(x) = \frac{1}{x^2 - 16} = (x^2 - 16)^{-1}$$

VA: $x^2 - 16 = 0$
 $x = \pm 4$

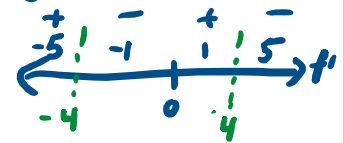
HA: $y = 0$



y-intercept: $y = \frac{1}{0 - 16} \Rightarrow (0, -1/16)$

x-intercept: $0 = \frac{1}{x^2 - 16} \Rightarrow \text{Never}$

$$y' = -(x^2 - 16)^{-2} (2x) = \frac{-2x}{(x^2 - 16)^2} = 0 \Rightarrow x = 0$$



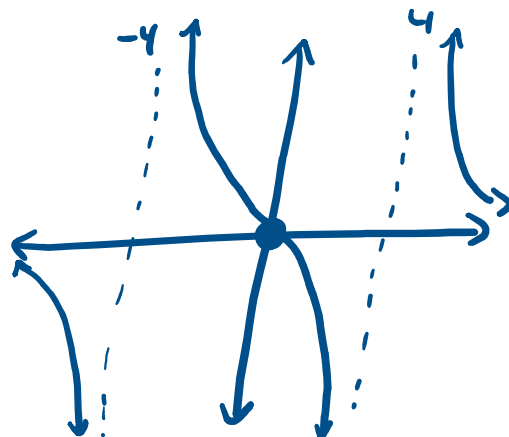
$$(9.1.2) \ y = f(x) = \frac{x}{x^2 - 16}$$

$$\underline{\text{VA: } x^2 - 16 = 0}$$

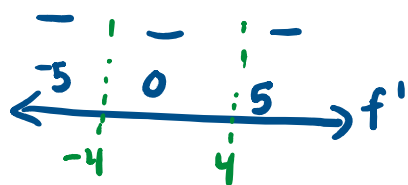
$$x = \pm 4$$

$$\underline{\text{y-intercept: } y = \frac{0}{0^2 - 16} = 0}$$

$$(0, 0)$$



$$f'(x) = \frac{1(x^2 - 16) - x(2x)}{(x^2 - 16)^2} = \frac{-x^2 - 16}{(x^2 - 16)^2} = 0 \text{ which never happens}$$



$$\lim_{x \rightarrow \infty} \frac{x}{x^2 - 16} = \lim_{x \rightarrow \infty} \frac{x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

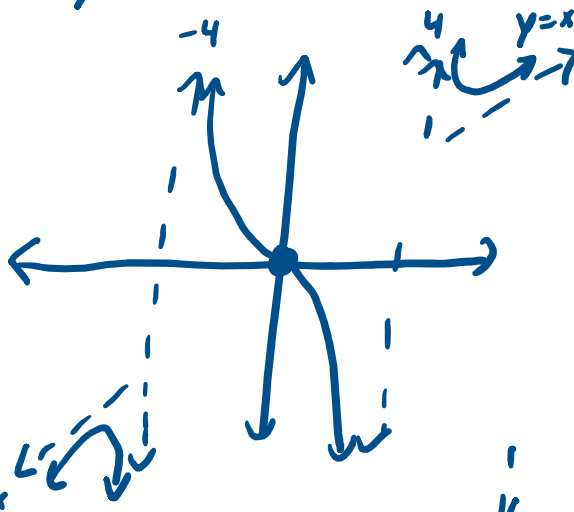
$$\lim_{x \rightarrow -\infty} \frac{x}{x^2 - 16} = \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$(9.1.3) \ y = f(x) = \frac{x^3}{x^2 - 16}$$

VA: $x^2 - 16 = 0$
 $x = \pm 4$

SA: $x^2 - 16 \overline{) \begin{matrix} x^3 + 0x^2 + 0x + 0 \\ -(x^3) \\ \hline 16x + 0 \end{matrix}}$

$y = x$ is SA



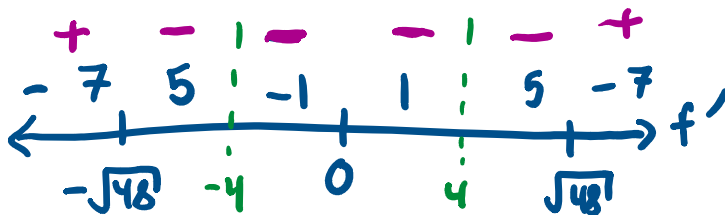
$$f'(x) = \frac{3x^2(x^2 - 16) - x^3(2x)}{(x^2 - 16)^2}$$

$$= \frac{x^2(3x^2 - 48 - 2x^2)}{(x^2 - 16)^2}$$

$$= \frac{x^2(x^2 - 48)}{(x^2 - 16)^2} = 0 \text{ if } x = 0, x = \pm\sqrt{48}$$

$$\sqrt{48} = 4\sqrt{3} \approx 6.9$$

$$\begin{array}{r} 16 \\ \times 3 \\ \hline 48 \\ 48 \\ \hline 4 \cdot 12 \end{array}$$



$$(9.1.4) \ y = f(x) = \frac{x}{x^2 + 16}$$

VA: None since $x^2 + 16 \neq 0$

y-intercept: $y = \frac{0}{0^2 + 16} = 0$

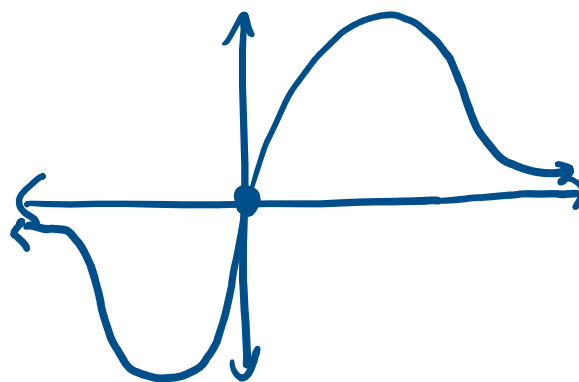
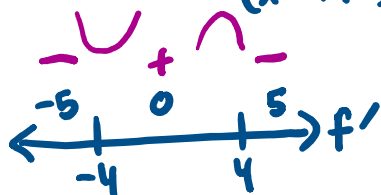
$(0, 0)$

End Behavior:

$$\lim_{x \rightarrow \infty} \frac{x}{x^2 + 16} = \lim_{x \rightarrow \infty} \frac{x}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{x}{x^2 + 16} = \lim_{x \rightarrow -\infty} \frac{x}{x^2} = \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$f'(x) = \frac{x^2 + 16 - x(2x)}{(x^2 + 16)^2} = \frac{16 - x^2}{(x^2 + 16)^2} = 0 \text{ if } x = \pm 4$$



$$(9.1.5) \ y = f(x) = -\frac{16x}{x^2 + 16}$$

VA: None since $x^2 + 16 \neq 0$

y-intercept: $y = \frac{-16 \cdot 0}{0^2 + 16} = 0$

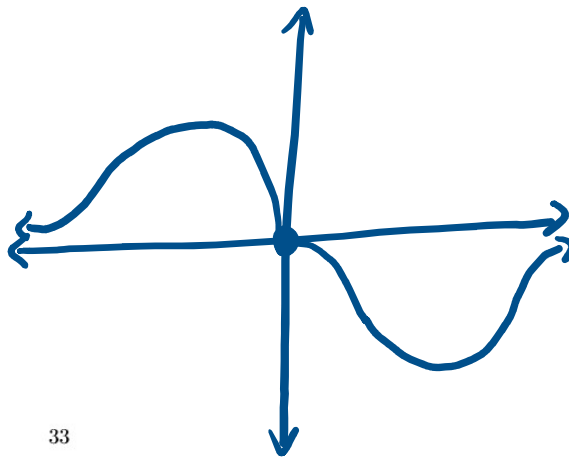
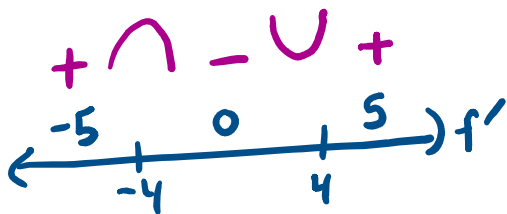
$(0, 0)$

End Behavior:

$$\lim_{x \rightarrow \infty} \frac{-16x}{x^2 + 16} = \lim_{x \rightarrow \infty} \frac{-16x}{x^2} = \lim_{x \rightarrow \infty} -\frac{16}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{-16x}{x^2 + 16} = \lim_{x \rightarrow -\infty} \frac{-16x}{x^2} = \lim_{x \rightarrow -\infty} -\frac{16}{x} = 0$$

$$f'(x) = \frac{-16(x^2 + 16 - x(2x))}{(x^2 + 16)^2} = \frac{-16(16 - x^2)}{(x^2 + 16)^2} = 0 \text{ if } x = \pm 4$$



$$(9.1.6) \ y = f(x) = \frac{x^3}{x^2+1}$$

VA: None since $x^2+1 \neq 0$

y-intercept: $y = \frac{0^3}{0^2+1} = 0$

$(0,0)$

SA: $x^2+1 \sqrt{x^3+0x^2+0x+0}$
 $-(x^3 + x) \downarrow$
 $-x+0$

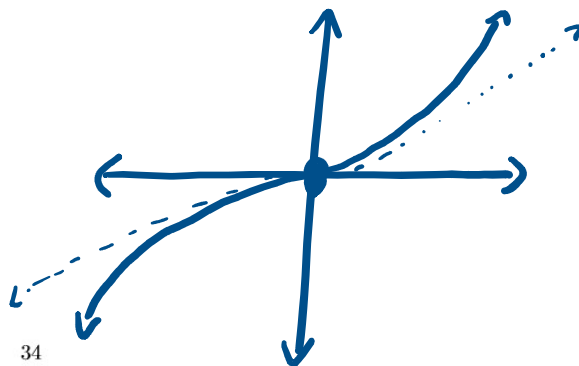
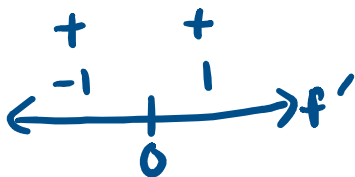
End Behavior:

$$\lim_{x \rightarrow \infty} \frac{x^3}{x^2+1} = \lim_{x \rightarrow \infty} \frac{x^3}{x^2} = \lim_{x \rightarrow \infty} x = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^3}{x^2+1} = \lim_{x \rightarrow -\infty} \frac{x^3}{x^2} = \lim_{x \rightarrow -\infty} x = -\infty$$

$$f'(x) = \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \frac{x^2(3x^2+3-2x^2)}{(x^2+1)^2}$$

$$= \frac{x^2(x^2+3)}{(x^2+1)^2} = 0 \quad \text{if } x=0$$



$$(9.1.7) \ y = f(x) = \frac{x^3}{1-x^2}$$

VA: $1-x^2=0$
 $x=\pm 1$

SA: $-x^2 + 1 \sqrt{x^3 + 0x^2 + 0x + 0}$
 $\frac{-(x^3 - x)}{+x+0}$
 $-x \rightarrow y=-x$

End Behavior:

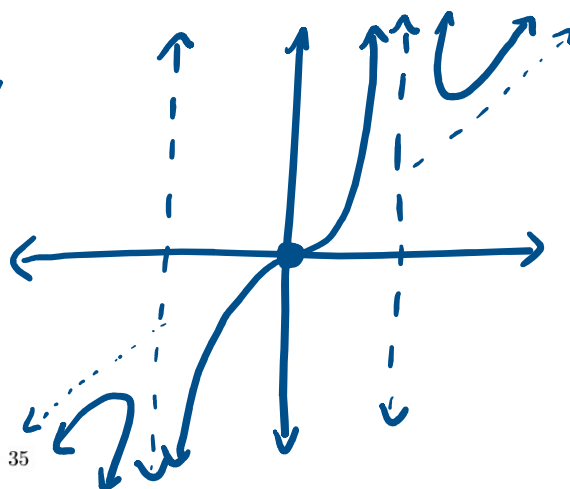
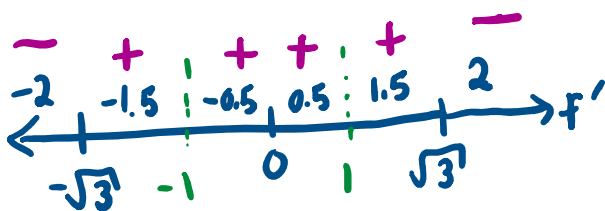
$$\lim_{x \rightarrow \infty} \frac{x^3}{1-x^2} = \lim_{x \rightarrow \infty} \frac{x^3}{-x^2} = \lim_{x \rightarrow \infty} -x = -\infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^3}{1-x^2} = \lim_{x \rightarrow -\infty} \frac{x^3}{-x^2} = \lim_{x \rightarrow -\infty} -x = \infty$$

$$\sqrt{3} \approx 1.7$$

$$f'(x) = \frac{3x^2(1-x^2) - x^3(-2x)}{(1-x^2)^2} = \frac{x^2(3-3x^2+2x^2)}{(1-x^2)^2}$$

$$= \frac{x^2(3-x^2)}{(1-x^2)^2} = 0 \text{ if } x=0, \pm\sqrt{3}$$



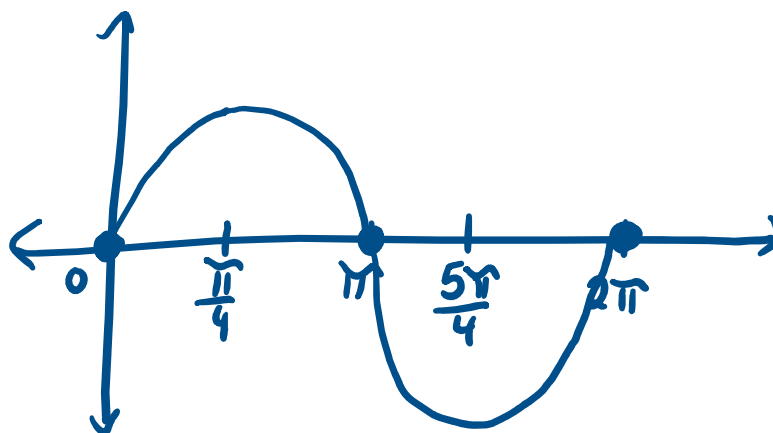
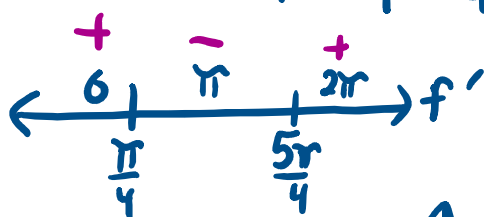
35

(9.1.8) $y = f(x) = e^{-x} \sin(x)$ on $[0, 2\pi]$

y-intercept: $y = e^0 \sin(0) = 0$
 $(0, 0)$

x-intercept: $0 = e^{-x} \sin(x)$
 $0 = \sin(x)$ b/c $e^{-x} \neq 0$
 $x = 0, \pi, 2\pi$ in $[0, 2\pi]$

$y' = -e^{-x} \sin(x) + e^{-x} \cos(x) = 0$
 $e^{-x}(-\sin(x) + \cos(x)) = 0$
 $\cos(x) = \sin(x)$
 $x = \frac{\pi}{4}, \frac{5\pi}{4}$



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$\ln((x-6)(x-4)) = \ln(x-6) + \ln(x-4)$

(9.1.9) $y = f(x) = \ln(x^2 - 10x + 24)$

Domain of $f(x)$

(9.1.9) $y = f(x) = \ln(x^2 - 10x + 24)$

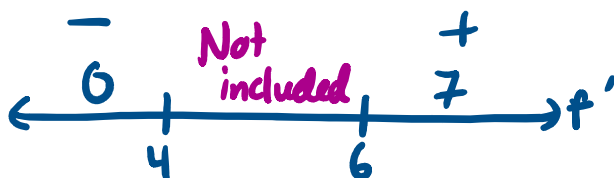
y-intercept: $y = \ln(0^2 - 0 + 24)$
 $(0, \ln(24))$

x-intercept: $0 = \ln(x^2 - 10x + 24)$
 $1 = x^2 - 10x + 24$
 $0 = x^2 - 10x + 24$
 $x = 5 \pm \sqrt{2}$

$y' = \frac{1}{x-6} + \frac{1}{x-4} = 0$
 $\frac{x-4 + x-6}{(x-6)(x-4)} = 0$
 $\frac{2x-10}{(x-6)(x-4)} = 0$

~~$2x-10=0$~~ $x-6=0$ $x-4=0$
 ~~$x=5$~~ $x=6$ $x=4$

B/c Domain of f



Domain of $f(x)$

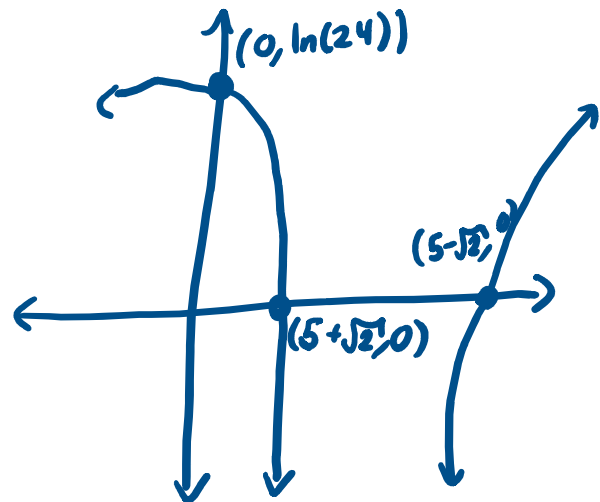
• Both $x-4 > 0$ and $x-6 > 0$

when $x > 6$

• Both $x-4 < 0$ and $x-6 < 0$

when $x-4 < 0$
 $x < 4$

So $(-\infty, 4) \cup (6, \infty)$



10. You are supposed to understand the idea of the linear approximation of a function $f(x)$ at $x = a$

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

and apply it to approximate the value of a function at a given point.

Example Problems:

(10.1) Find the formula for the linear approximation to $f(x) = \sqrt{x}$ at $x = a = 1$ and use it to approximate $\sqrt{1.1}$.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(1) = 1$$

$$f'(1) = \frac{1}{2}$$

$$f(x) = f(1) + f'(1)(x-1)$$

$$= 1 + \frac{1}{2}(x-1)$$

$$f(1.1) = 1 + \frac{1}{2}(1.1-1)$$

$$= 1 + \frac{1}{2}(0.1)$$

$$= 1 + \frac{1}{2} \cdot \frac{1}{10}$$

$$= 1 + \frac{0.5}{10} = 1 + \frac{5}{100}$$

$$\sqrt{1.1} \approx 1.05$$

(10.2) Find the formula for the linear approximation to $f(x) = e^x$ at $x = a = 0$ and use it to approximate $e^{-0.05}$.

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$f(x) = f(0) + f'(0)(x-0)$$

$$= 1 + 1 \cdot (x-0)$$

$$= 1 + x$$

$$f(-0.05) = 1 - 0.05$$

$$= 0.95$$

$$e^{-0.05} \approx 0.95$$

(10.3) Find the estimate of $\sqrt[3]{8.012}$ using the linear approximation of the function $f(x) = \sqrt[3]{x}$ at $x = 8$.

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3 \cdot (x^{2/3})}$$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}$$

$$f(x) = f(8) + f'(8)(x-8)$$

$$= 2 + \frac{1}{12}(x-8)$$

$$= 2 + \frac{1}{12}(8.012-8)$$

$$= 2 + \frac{1}{12} \cdot 0.012$$

$$= 2 + \frac{12}{12000}$$

$$= 2 + \frac{1}{1000}$$

$$\sqrt[3]{8.012} \approx 2.001$$

(10.4) Find the estimate of $\sin(46^\circ)$ using the linear approximation of the function $f(x) = \sin(x)$ at $a = \pi/4$.

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f(\pi/4) = \sqrt{2}/2$$

$$f'(\pi/4) = \sqrt{2}/2$$

$$f(x) = f(\pi/4) + f'(\pi/4)(x - \pi/4)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4)$$

$$f(\frac{46\pi}{180}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(\frac{46\pi}{180} - \frac{45\pi}{180})$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\pi}{180}$$

$$46^\circ = 46^\circ \cdot \frac{\pi}{180} = \frac{46\pi}{180}$$

$$\sin(46^\circ) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{360}$$

(10.5) Find the estimate of $\sin(29^\circ)$ using the linear approximation of the function $f(x) = \sin(x)$ at $a = \pi/6$.

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)(x - \frac{\pi}{6})$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(\frac{29\pi}{180} - \frac{30\pi}{180}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(-\frac{\pi}{180}\right)$$

$$29^\circ = 29^\circ \cdot \frac{\pi}{180^\circ} = \frac{29\pi}{180}$$

$$\sin(29^\circ) \approx \frac{1}{2} - \frac{\sqrt{3}\pi}{360}$$

(10.6) Find the estimate of $\sin(61^\circ)$ using the linear approximation of the function $f(x) = \sin(x)$ at $a = \pi/3$.

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)(x - \frac{\pi}{3})$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2} \left(x - \frac{\pi}{3}\right)$$

$$f(61^\circ) = \frac{\sqrt{3}}{2} + \frac{1}{2} \left(\frac{61\pi}{180} - \frac{60\pi}{180}\right)$$

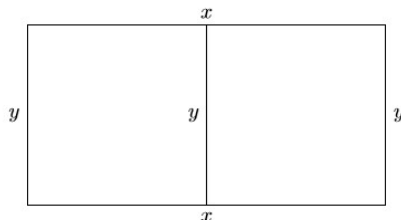
$$= \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\pi}{180}$$

$$61^\circ = 61^\circ \cdot \frac{\pi}{180^\circ} = \frac{61\pi}{180}$$

$$\sin(61^\circ) \approx \frac{\sqrt{3}}{2} + \frac{\pi}{360}$$

11. Total of 3 Optimization Problems will be given in Exam 3 in PIN.

- (11.1) A farmer has 400 feet of fencing to build a rectangular pen to contain chickens. The pen needs to be in shape of a rectangle with a straight divider in the middle that separates the pen into two congruent rectangles. What is the maximum possible area inside the pen?



$$\begin{aligned} 400 &= P = 2x + 3y \\ 400 - 3y &= 2x \\ \frac{400 - 3y}{2} &= x \end{aligned}$$

$$\begin{aligned} A &= xy \\ &= \left(\frac{400}{2} - \frac{3y}{2} \right) y \\ &= 200y - \frac{3y^2}{2} \end{aligned}$$

$$\begin{aligned} A' &= 200 - 3y = 0 \\ 200 &= 3y \\ y &= \frac{200}{3} \end{aligned}$$

Find x.

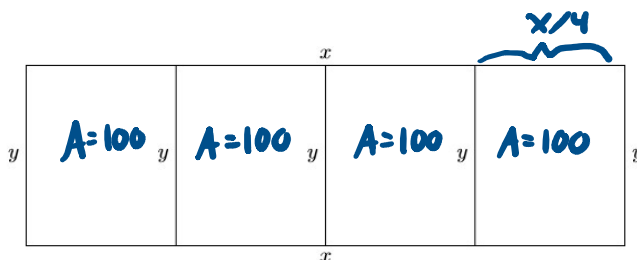
$$400 = 2x + 3\left(\frac{200}{3}\right)$$

$$\begin{aligned} 400 &= 2x + 200 \\ 200 &= 2x \\ x &= 100 \end{aligned}$$

$$A = 100 \left(\frac{200}{3} \right) = \frac{20000}{3}$$

Answer: _____

- (11.2) A farmer plans to make four identical and adjacent rectangular pens against a barn. Each with an area of 100 m^2 what are the dimensions of each pen that minimize the amount of fence that must be used? (Consider the sides as the lengths as x and y with y being the side parallel to the barn.)



$$P = 2x + 5y$$

$$P = 2(400y^{-1}) + 5y$$

$$= 800y^{-1} + 5y$$

$$P' = -800y^{-2} + 5 = 0$$

$$-\frac{800}{y^2} + 5 = 0$$

$$\frac{5}{1} = \frac{800}{y^2}$$

$$5y^2 = 800$$

$$y^2 = 160$$

$$y = 4\sqrt{10}$$

$$100 = y \left(\frac{x}{4} \right)$$

$$400 = xy$$

$$400y^{-1} = x$$

Find x .

$$400 = xy \text{ w/ } y = 4\sqrt{10}$$

$$400 = 4\sqrt{10}x$$

$$100 = \sqrt{10}x$$

$$\frac{100}{\sqrt{10}} = x$$

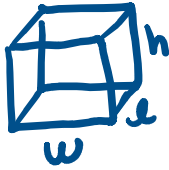
$$\frac{100\sqrt{10}}{10} = x$$

$$10\sqrt{10} = x$$

Dimensions $10\sqrt{10}$ by $4\sqrt{10}$

Answer: _____

- (11.3) A rectangular box has to be made with the width being twice as long as the length (with a bottom but WITHOUT a top). If the surface area of the box is 400 cm^2 , what is the height of the box with the largest volume?



$$w = 2l$$

$$\begin{aligned} V &= w \cdot l \cdot h \\ &= (2l) \cdot l \cdot h \\ &= 2l^2 h \end{aligned}$$

$$V = 2l^2 \left(\frac{200 - l^2}{3l} \right)$$

$$= \frac{2}{3} l (200 - l^2)$$

$$= \frac{2}{3} [200l - l^3]$$

$$V' = \frac{2}{3} [200 - 3l^2] = 0$$

$$200 - 3l^2 = 0$$

$$200 = 3l^2$$

$$\frac{200}{3} = l^2$$

$$\sqrt{\frac{200}{3}} = l$$

$$400 = A = wl + 2lh + 2wh$$

$$400 = (2l)l + 2lh + 2(2l)h$$

$$400 = 2l^2 + 2lh + 4lh$$

$$400 = 2l^2 + 6lh$$

$$200 = l^2 + 3lh$$

$$200 - l^2 = 3lh$$

$$\frac{200 - l^2}{3l} = h$$

Find h .

$$h = \frac{200 - \left(\frac{200}{3}\right)}{3\left(\sqrt{\frac{200}{3}}\right)}$$

$$= \frac{2 \cdot 200}{3}$$

$$\frac{2\sqrt{600}}{3}$$

$$= \frac{400}{3\sqrt{600}}$$

$$= \frac{400\sqrt{6} \cdot 10}{3 \cdot 600}$$

Answer: $\frac{40\sqrt{6}}{18} = \frac{20\sqrt{6}}{9}$

(11.4) A cylindrical barrel is to be made that has a volume of $6\pi \text{ ft}^3$. The material for the top and bottom cost \$ 4 per square foot and the material for the side costs \$ 6 per square foot. Find the radius of the barrel that will minimize the cost of production.



$$\begin{aligned} V &= \pi r^2 h \\ 6\pi &= \pi r^2 h \\ 6 &= r^2 h \\ \frac{6}{r^2} &= h \end{aligned}$$

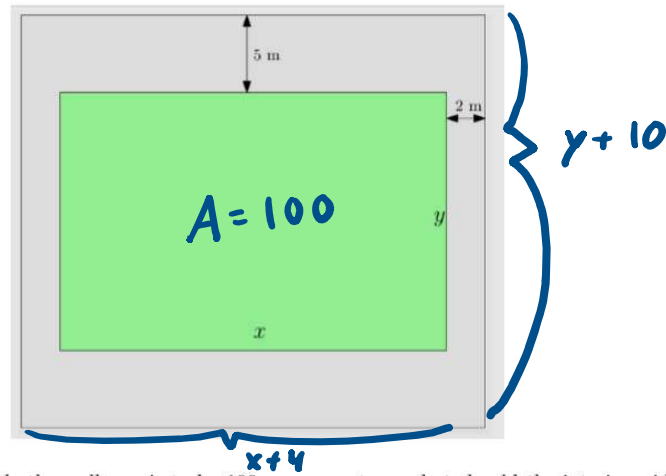
$$\begin{aligned} C &= 4(2\pi r) + 6(2\pi r h) \\ &= 8\pi r + 12\pi r h \\ &= 8\pi r + 12\pi r \left(\frac{6}{r^2}\right) \\ &= 8\pi r + 72\pi r^{-1} \end{aligned}$$

$$\begin{aligned} C' &= 8\pi - 72\pi r^{-2} = 0 \\ 8\pi &= \frac{72\pi}{r^2} \end{aligned}$$

$$\begin{aligned} 8\pi r^2 &= 72\pi \\ r^2 &= 9 \\ r &= 3 \end{aligned}$$

Answer: _____

- (11.5) Suppose a rectangular area is to be surrounded by a concrete walkway that is 2 meter wide on the East and West and 5 meters wide on the North and South.



If the area inside the walkway is to be 100 square meters, what should the interior width of the enclosed area be (labeled x in the diagram above) in order to minimize the amount of concrete used.

$$C = (x+4)(y+10) - xy$$

$$100 = xy$$

$$= \left(\frac{100}{y} + 4\right)(y+10) - 100$$

$$\frac{100}{y} = x$$

$$= \cancel{100} + 4y + 40 + \frac{1000}{y} - \cancel{100}$$

$$= 4y + 40 + 1000y^{-1}$$

$$C' = 4 - 1000y^{-2} = 0$$

$$4 = \frac{1000}{y^2}$$

$$4y^2 = 1000$$

$$y^2 = 250$$

$$y = 5\sqrt{10}$$

Find x .

$$x = \frac{100}{5\sqrt{10}} = \frac{100\sqrt{10}}{5 \cdot 10} = 2\sqrt{10}$$

Answer: _____

$$2\sqrt{10}$$

- (11.6) Liz is standing on the bank of 3 km wide river. An ice cream shop is located on the opposite bank 10 km down the river from the point right across Liz. Liz plans to get to the ice cream shop by a combination of swimming across the river to a point x km down river on the opposite bank, and then jogging the rest of the way along the bank. She swims at a rate of 2 km/hour and jogs at a rate of 3 km/hour. Find the point where Liz should reach by swimming on the opposite bank in order for her to get to the ice cream shop in the least amount of time.

then jogging the rest of the way along the bank. She swims at a rate of 2 km/hour and jogs at a rate of 3 km/hour. Find the point where Liz should reach by swimming on the opposite bank in order for her to get to the ice cream shop in the least amount of time.



$$T(x) = \frac{\sqrt{x^2 + 9}}{2} + \frac{10 - x}{3}$$

$$T'(x) = \frac{\frac{1}{2}(x^2 + 9)^{-\frac{1}{2}}(2x)}{2} + \frac{-1}{3}$$

$$= \frac{x}{2\sqrt{x^2 + 9}} - \frac{1}{3} = 0$$

$$\frac{x}{2\sqrt{x^2 + 9}} = \frac{1}{3}$$

$$3x = 2\sqrt{x^2 + 9}$$

$$(3x)^2 = (2\sqrt{x^2 + 9})^2$$

$$9x^2 = 4(x^2 + 9)$$

$$9x^2 = 4x^2 + 36$$

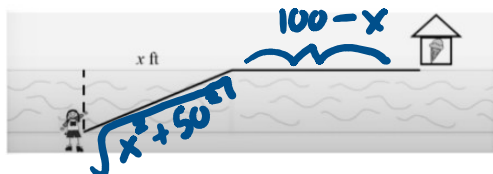
$$5x^2 = 36$$

$$x^2 = \frac{36}{5}$$

$$x = \sqrt{\frac{36}{5}} = \frac{6\sqrt{5}}{5}$$

Answer: $\frac{6\sqrt{5}}{5}$

- (11.7) Liz is standing on the bank of a 50-foot wide river. An ice cream shop is located 100 feet down river on the opposite bank. Liz plans to get to the ice cream shop by a combination of swimming across the river to a point x feet down river on the opposite bank, and then jogging the rest of the way along the bank. She swims at a rate of 4 ft/s and jogs at a rate of 5 ft/s. How many feet downstream of Liz on the opposite bank should she swim to reach the ice cream shop in the least amount of time?



$$T(x) = \frac{\sqrt{x^2 + 2500}}{4} + \frac{100 - x}{5}$$

$$T'(x) = \frac{\frac{1}{2}(x^2 + 2500)^{-1/2}(2x)}{4} + \frac{-1}{5}$$

$$= \frac{x}{4\sqrt{x^2 + 2500}} - \frac{1}{5} = 0$$

$$\frac{x}{4\sqrt{x^2 + 2500}} = \frac{1}{5}$$

$$5x = 4\sqrt{x^2 + 2500}$$

$$(5x)^2 = (4\sqrt{x^2 + 2500})^2$$

$$25x^2 = 16(x^2 + 2500)$$

$$25x^2 = 16x^2 + 16 \cdot 2500$$

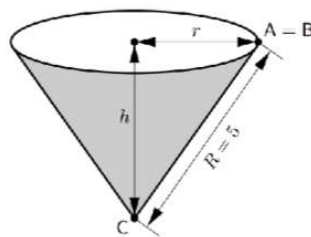
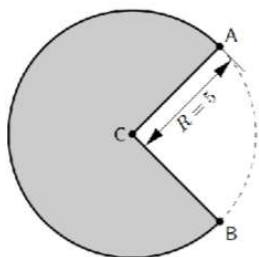
$$9x^2 = 16 \cdot 2500$$

$$x^2 = \frac{16 \cdot 2500}{9}$$

$$x = \frac{4 \cdot 50}{3}$$

Answer: 200/3

- (11.8) A cone-shaped drinking cup is made from a circular piece of paper of fixed radius $R = 5$ by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



HINT: The volume V of a right circular cone with radius r and height h as illustrated in the picture above is given by

$$V = \frac{1}{3} \pi r^2 h$$

$$h^2 + r^2 = 5^2$$

$$r^2 = 25 - h^2$$

$$r^2 = 25 - \frac{25}{3}$$

$$r^2 = \frac{50}{3}$$

$$V = \frac{1}{3} \pi (25 - h^2) h$$

$$= \frac{\pi}{3} (25h - h^3)$$

$$V' = \frac{\pi}{3} (25 - 3h^2) = 0$$

$$25 = 3h^2$$

$$\frac{25}{3} = h^2$$

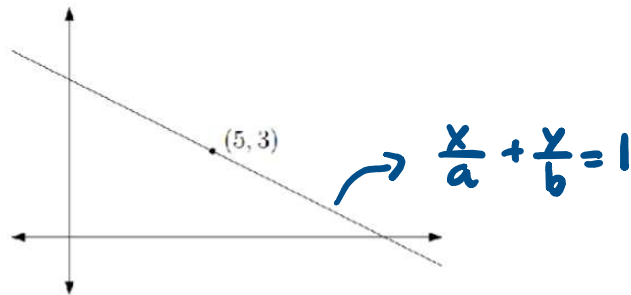
$$\sqrt{\frac{25}{3}} = h$$

$$V = \frac{\pi}{3} \left(\frac{50}{3} \right) \left(\frac{5}{\sqrt{3}} \right)$$

$$= \frac{250}{9\sqrt{3}} \pi$$

Answer: _____

(11.9) Find an equation of the line through the point (5,3) that cuts off the least area from the first quadrant.



This line needs to pass through (5,3). So

$$\frac{5}{a} + \frac{3}{b} = 1$$

$$5b + 3a = ab$$

$$3a = ab - 5b$$

$$3a = b(a-5)$$

$$\frac{3a}{a-5} = b$$

$$A = \frac{1}{2}ab$$

$$= \frac{1}{2}a \left(\frac{3a}{a-5} \right)$$

$$= \frac{3}{2} \left(\frac{a^2}{a-5} \right)$$

$$A' = \frac{3}{2} \left(\frac{2a(a-5) - a^2}{(a-5)^2} \right)$$

$$= \frac{3}{2} \left(\frac{2a^2 - 10a - a^2}{(a-5)^2} \right)$$

$$= \frac{3}{2} \left(\frac{a^2 - 10a}{(a-5)^2} \right) = 0$$

$$a^2 - 10a = 0$$

$$a(a-10) = 0$$

$$a \neq 0, a = 10$$



b/c then

$\frac{x}{a} + \frac{y}{b} = 1$ will have a 0 in denominator

$$b = \frac{3 \cdot 10}{10-5} = \frac{30}{5} = 6$$

So the eqn is

$$\frac{x}{10} + \frac{y}{6} = 1$$

$$6x + 10y = 60$$

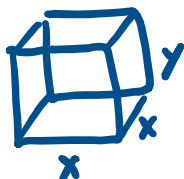
$$10y = -6x + 60$$

$$y = -\frac{6}{10}x + \frac{60}{10}$$

$$y = -\frac{3}{5}x + 6$$

- (11.10) Matt is building an open top wooden box for his cat Rupert, who likes to play with boxes. Matt is using wood on all four sides and the bottom, and is covering the wood on the bottom with carpet.

Matt would like the bottom to have a square base. The cost of wood is \$ 2 per ft^2 and the cost of the cost of the carpet is \$ 1 ft^2 . Rupert requires 48 ft^3 to have a satisfactory play in the box. What is the minimum total cost of this box.



$$V = x^2 y$$

$$48 = x^2 y$$

$$48x^{-2} = y$$

$$C = 2(x^2 + 4xy) + 1(x^2)$$

$$= 3x^2 + 8xy$$

$$= 3x^2 + 8x(48x^{-2})$$

$$= 3x^2 + 8 \cdot 48x^{-1}$$

$$C' = 6x - 8 \cdot 48x^{-2} = 0$$

$$\frac{6x}{1} = \frac{8 \cdot 48}{x^2}$$

$$6x^3 = 8 \cdot 48$$

$$x^3 = 8 \cdot 8$$

$$x = 2 \cdot 2 = 4$$

$$C(4) = 3(4)^2 + \frac{8 \cdot 48}{4}$$

$$= 3 \cdot 16 + 2 \cdot 48 = \$144$$

Answer: _____

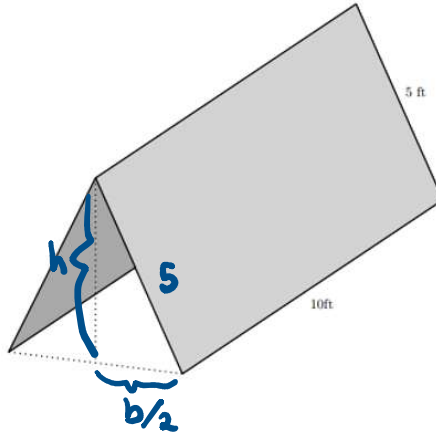
- (11.11) A Purdue student goes camping with a square tarp, each side measuring 10 feet long. Since the sky get darker, he decides to make a storage place by folding the tarp in the middle as shown in the picture to avoid the rain. What is the largest volume of the storage place he can create underneath the tarp?

$$5^2 = h^2 + \left(\frac{b}{2}\right)^2$$

$$25 = h^2 + \frac{b^2}{4}$$

$$100 = 4h^2 + b^2$$

$$\sqrt{100 - 4h^2} = b$$



Find b with $h^2 = \frac{25}{2}$

$$\begin{aligned} b &= \sqrt{100 - 4h^2} \\ &= \sqrt{100 - 4\left(\frac{25}{2}\right)} \\ &= \sqrt{100 - 50} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} V &= 5(5\sqrt{2})\left(\frac{5}{\sqrt{2}}\right) \\ &= \boxed{125} \end{aligned}$$

$$V = (\text{area of triangle}) \times \text{length}$$

$$= \frac{1}{2} b h (10)$$

$$= 5 b h$$

$$= 5 h \sqrt{100 - 4h^2}$$

$$V' = 5\sqrt{100 - 4h^2} + 5h \cdot \frac{1}{2}(100 - 4h^2)^{-1/2}(-8h)$$

$$5\sqrt{100 - 4h^2} - \frac{20h^2}{\sqrt{100 - 4h^2}} = 0$$

$$5\sqrt{100 - 4h^2} = \frac{20h^2}{\sqrt{100 - 4h^2}}$$

$$5(100 - 4h^2) = 20h^2 \quad \left| \quad \frac{25}{2} = h^2 \right.$$

$$100 - 4h^2 = 4h^2$$

$$100 = 8h^2$$

$$h = \frac{5}{\sqrt{2}}$$

Answer: _____

(11.12) What is the largest area of the rectangle inscribed in an ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$A = 4xy$$

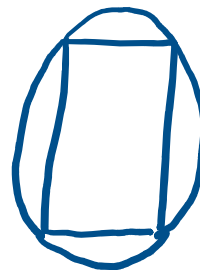
$$A = \frac{4x}{9} \sqrt{36 - 4x^2}$$

$$4x^2 + 9y^2 = 36$$

$$9y^2 = 36 - 4x^2$$

$$y^2 = \frac{36 - 4x^2}{9}$$

$$y = \frac{\sqrt{36 - 4x^2}}{3}$$



$$A' = \frac{4}{9} \left(\sqrt{36 - 4x^2} + \frac{x}{2} (36 - 4x^2)^{-1/2} (-8x) \right) = 0$$

$$\sqrt{36 - 4x^2} - \frac{4x^2}{\sqrt{36 - 4x^2}} = 0$$

$$\sqrt{36 - 4x^2} = \frac{4x^2}{\sqrt{36 - 4x^2}}$$

$$36 - 4x^2 = 4x^2$$

$$36 = 8x^2$$

$$\frac{9}{2} = x^2$$

$$\frac{3}{\sqrt{2}} = x$$

$$y = \frac{\sqrt{36 - 4\left(\frac{3}{\sqrt{2}}\right)^2}}{3}$$

$$= \frac{\sqrt{36 - 2 \cdot 9}}{3}$$

$$= \frac{\sqrt{36 - 18}}{3}$$

$$= \frac{3\sqrt{2}}{3}$$

$$= \sqrt{2}$$

$$V = 4\left(\frac{3}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{1}\right) = 12$$

Answer: _____