Please show **all** your work! Answers without supporting work will not be given credit.
Write answers in spaces provided.

Name:____

New materials after Exam 3

Remark: There will be 5 questions regarding the new materials after Exam 3 in the Final Exam. Look also at the "NOTE" for the subject matter 14 about computing the total distance.

1. You are supposed to know how to compute the integral using the Riemann Sum. Conversely, you should know how to compute the limit in the form of Riemann Sum using the integration and Fundamental Theorem of Calculus. You are also supposed to know how to compute the integral knowing its geometrical meaning.

   Example Problems

   (1.1)(1.1.i) Write down the formula for approximating the integration $\int_0^1 \sqrt{1-x^2}\,dx$ as the Riemann Sum dividing the interval $[0,1]$ into n equal subintervals and using the right end points.

$$\int_0^1 \sqrt{1-x^2}\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

   (1.1.ii) Compute the integral using the geometric interpretation.

$$\int_0^1 \sqrt{1-x^2}\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(1.2) Evaluate the following definite integrals

(1.2.1) $\displaystyle\int_3^5 \sqrt{16-(x-3)^2}\,dx$

$$\int_3^5 \sqrt{16-(x-3)^2}\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(1.2.2) $\int_1^5 (1+\sqrt{4-(x-3)^2})\,dx$

$$\int_1^5 (1+\sqrt{4-(x-3)^2})\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(1.2.3) $\int_{-1}^{3+2\sqrt{2}} \sqrt{16-(x-3)^2}\,dx$

$$\int_{-1}^{3+2\sqrt{2}} \sqrt{16-(x-3)^2}\,dx = \underline{\hspace{4cm}}$$

(1.2.4) $\int_{-1}^{1} (1+\sqrt{1-x^2})\,dx$

$$\int_{-1}^{1} (1+\sqrt{1-x^2})\,dx = \underline{\hspace{5cm}}$$

(1.2.5) $\int_0^{\sqrt{3}} (1 + \sqrt{4 - x^2})\, dx$

$\int_0^{\sqrt{3}} (1 + \sqrt{4 - x^2})\, dx$ ____________________

(1.2.6) $\int_{-3}^{9} \sqrt{36 - (x - 3)^2}\, dx$

$\int_{-3}^{9} \sqrt{36 - (x - 3)^2}\, dx =$ ____________________

(1.2.7) $\int_1^5 (1 + \sqrt{4-(x-3)^2})\,dx$

$$\int_1^5 (1 + \sqrt{4-(x-3)^2})\,dx = \underline{\hspace{4cm}}$$

(1.3) Evaluate the following definite integral $\int_0^5 |3x-1|\,dx$

$$\int_0^5 |3x-1|\,dx = \underline{\hspace{6cm}}$$

(1.4) Compute the following limits

(1.4.1) $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{n^2+k^2}\right)$

HINT: Use the equation

$$\sum_{k=1}^{n}\frac{k}{n^2+k^2}=\sum_{k=1}^{n}\frac{k/n}{1+(k/n)^2}\cdot\frac{n}{n^2}=\sum_{k=1}^{n}\frac{k/n}{1+(k/n)^2}\cdot\frac{1}{n}=\sum_{k=1}^{n}f(x_k^*)\Delta x$$

where

$$\begin{cases} f(x)=\dfrac{x}{1+x^2} \\ x_k^*=k/n \\ \Delta x=\dfrac{1}{n}=b-an \end{cases}$$

Now

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k^*)\Delta x=\int_0^1 f(x)\,dx$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{n^2+k^2}\right)= \qquad \underline{\hspace{6cm}}$$

(1.4.2) $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n+4k}\right)$

HINT: Use the equation

$$\sum_{k=1}^{n}\frac{1}{n+4k}=\sum_{k=1}^{n}\frac{1}{1+4(k/n)}\cdot\frac{1}{n}=\sum_{k=1}^{n}f(x_k^*)\Delta x$$

where

$$\begin{cases} f(x)=\dfrac{1}{1+4x} \\ x_k^*=k/n \\ \Delta x=\dfrac{1}{n}=b-an \end{cases}$$

Now

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k^*)\Delta x=\int_0^1 f(x)\,dx$$

$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n+4k}\right)=$ ____________________

(1.4.3) $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{n}{n^2+k^2}\right)$

HINT: Use the equation

$$\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\sum_{k=1}^{n}\frac{1}{1+(k/n)^2}\cdot\frac{n}{n^2}=\sum_{k=1}^{n}\frac{1}{1+(k/n)^2}\cdot\frac{1}{n}=\sum_{k=1}^{n}f(x_k^*)\Delta x$$

where

$$\begin{cases} f(x)=\dfrac{1}{1+x^2} \\ x_k^*=k/n \\ \Delta x=\dfrac{1}{n}=b-an \end{cases}$$

Now

$$\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k^*)\Delta x=\int_0^1 f(x)\,dx$$

$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{n}{n^2+k^2}\right)=$ ______________________

(1.4.4) $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} \right)$

HINT: Use the equation

$$\sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} = \sum_{k=1}^{n} \frac{1}{\sqrt{1 - (k/2n)^2}} \cdot \frac{1}{2n} = \sum_{k=1}^{n} \frac{1}{\sqrt{1 - \left(k \cdot \frac{1/2}{n}\right)^2}} \cdot \frac{1/2}{n} = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

where

$$\begin{cases} f(x) = \dfrac{1}{\sqrt{1-x^2}} \\ x_k^* = k \cdot \dfrac{1/2}{n} \\ \Delta x = \dfrac{1/2}{n} = b - an \end{cases}$$

Now

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^{1/2} f(x)\, dx = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} \right) =$ ____________________

(1.4.5) $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt{n^2-k^2}}\right)$

$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt{n^2-k^2}}\right) =$ ________________

2. You are supposed to understand the meaning of the Fundamental Theorem of Calculus, and use it to compute the derivative of a function given in the form of an integration. (This includes the requirement that you are supposed to know how to take an anti-derivative of a function.) You are also supposed to understand the special features of the integration involving even/odd functions.

Example Problems

(2.1) Set $F(x) = \displaystyle\int_x^{x^2} \sin(\pi t)\,dt$. Compute $F'(1/2)$.

$F'(1/2) =$ ____________________

(2.2) Set $F(x) = \displaystyle\int_x^{x^2} \cos(\pi t)\,dt$. Compute $F'(1/2)$.

$F'(1/2) =$ ____________________

(2.3) Compute the following.

(2.3.1) $\dfrac{d}{dx}\left(\displaystyle\int_0^{x^4} \sec(t)\,dt\right)$

$$\frac{d}{dx}\left(\int_0^{x^4} \sec(t)\,dt\right) = \underline{\hspace{6cm}}$$

(2.3.2) $\dfrac{d}{dx}\left(\displaystyle\int_{2x}^{3x} \frac{t^2-1}{t^2+1}\,dt\right)$

$$\frac{d}{dx}\left(\int_{2x}^{3x} \frac{t^2-1}{t^2+1}\,dt\right) = \underline{\hspace{6cm}}$$

(2.3.3) $\dfrac{d}{dx}\left(\displaystyle\int_{x}^{4x^2} \ln(t)\,dt\right)$

$$\frac{d}{dx}\left(\int_{x}^{4x^2} \ln(t)\,dt\right) = \underline{\hspace{5cm}}$$

(2.3.4) $\dfrac{d}{dx}\left(\displaystyle\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln(t)\,dt\right)$

$$\frac{d}{dx}\left(\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln(t)\,dt\right) = \underline{\hspace{5cm}}$$

(2.3.5) $\dfrac{d}{dx}\left(\displaystyle\int_{\sqrt{x}}^{x^2} \frac{e^t}{t}\,dt\right)$

$$\frac{d}{dx}\left(\int_{\sqrt{x}}^{x^2} \frac{e^t}{t}\,dt\right) = \rule{4cm}{0.4pt}$$

(2.4) Compute the following integrals.

(2.3.1) $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2(x)\,dx$

$$\int_{\pi/4}^{\pi/3} \sec^2(x)\,dx \rule{4cm}{0.4pt}$$

(2.3.2) $\int_0^4 2^x\,dx$

$\int_0^4 2^x\,dx =$ ______________________

(2.3.3) $\int_{-2}^2 (1-|x|^3)\,dx$

$\int_{-2}^2 (1-|x|^3)\,dx =$ ______________________

(2.3.4) $\displaystyle\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}\,dx$

$$\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}\,dx = \underline{\hspace{4cm}}$$

3. You are supposed to know how to use the Substitution Rule to compute the indefinite and/or definite integrals.

   Example Problems

   (3.1) Compute the following integrals:

   (3.1.1) $\displaystyle\int \tan(x)\,dx$

$$\int \tan(x)\,dx = \underline{\hspace{4cm}}$$

(3.1.2) $\displaystyle\int \frac{\ln(x)}{x}\,dx$

$$\int \frac{\ln(x)}{x}\,dx = \underline{\hspace{6cm}}$$

(3.1.3) $\displaystyle\int_{-1}^{0} x\sqrt{x+1}\,dx$

$$\int_{-1}^{0} x\sqrt{x+1}\,dx = \underline{\hspace{6cm}}$$

(3.1.4) $\int_0^2 x^5\sqrt{1+x^2}\,dx$

$$\int_0^2 x^5\sqrt{1+x^2}\,dx = \underline{\hspace{5cm}}$$

(3.1.5) $\int_0^{\pi/4} \sec^4(x)\tan(x)\,dx$

$$\int_0^{\pi/4} \sec^4(x)\tan(x)\,dx = \underline{\hspace{5cm}}$$

(3.1.6) $\int_0^{\pi/4} \sqrt{1+\tan(x)} \cdot (\sec^2(x))\, dx$

$$\int_0^{\pi/4} \sqrt{1+\tan(x)} \cdot (\sec^2(x))\, dx = \underline{\qquad\qquad}$$

(3.1.7) $\int_0^{\pi/4} (1+e^{\tan(x)}) \cdot (\sec^2(x))\, dx$

$$\int_0^{\pi/4} (1+e^{\tan(x)}) \cdot (\sec^2(x))\, dx = \underline{\qquad\qquad}$$

(3.1.8) $\int_{\pi/2}^{\pi}(1-\sin^2(x))\cos(x)\,dx$

$$\int_{\pi/2}^{\pi}(1-\sin^2(x))\cos(x)\,dx = \underline{\hspace{4cm}}$$

(3.1.9) $\int_{0}^{\pi/2}\frac{\sin(x)}{1+\cos^2(x)\,dx}$

$$\int_{0}^{\pi/2}\frac{\sin(x)}{1+\cos^2(x)\,dx} = \underline{\hspace{4cm}}$$

(3.1.10) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \underline{\hspace{6cm}}$$

(3.1.11) $\displaystyle\int_{-\ln(2)}^{(\ln(3)-\ln(4))/2} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

$$\int_{-\ln(2)}^{(\ln(3)-\ln(4))/2} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \underline{\hspace{4cm}}$$

(3.1.12) $\int_0^1 \frac{\tan^{-1}(x)}{1+x^2}\,dx$

$$\int_0^1 \frac{\tan^{-1}(x)}{1+x^2}\,dx = \rule{5cm}{0.4pt}$$

(3.1.13) $\int_e^{e^2} \frac{\ln(x^4)}{x}\,dx$

$$\int_e^{e^2} \frac{\ln(x^4)}{x}\,dx = \rule{5cm}{0.4pt}$$

4. You are supposed to understand that the differential equation

$$\frac{dy}{dt} = ky$$

has a solution of the form $y = y(0)e^{kt}$, and should be able to apply it to analyze the population growth and radioactive decay. In the case of the radioactive decay, you should also understand the formula $m(t) = m(0)2^{-t/h}$ in terms of the half-life h.

Example Problems

(4.1) A culture of a single cell creature Amoeba is found to triple its population in three weeks. Find its relative growth rate k. The formula for the population $P(t)$ as a function of time t is given as follows using the relative growth rate: $P(t) = P(0)e^{kt}$.

$k =$ ______________________

(4.2) The number of bacteria in a cell culture is initial observed to be 50. Three hours later the number is 100. Assuming that the bacteria grow exponentially, how many hours after the initial observation does the number of bacteria become equal to 700?

$t =$ ______________________

(4.3) The half-life of cesium-137 is 30 years. Suppose we have a 120-mg sample at the beginning. How long will it take until the remain of the sample becomes 2-mg?

Answer:____________________ $t =$

(4.4) Initially there was 20 gm of a radioactive substance. After 5 hours, only 4 gm remained. What is the half-life of this substance?

half-life = ____________________

(4.5) Suppose that 80% of a radioactive substance decays in 5 hours. Find the half-life of the radioactive substance.

half-life = ____________________

(4.6) The population of a town with population 90,000 in year 2016 grows at a rate of 2.4 % per year. After how many years will the population reach 120,000?

$t =$ ____________________

(4.7) The radioactive isotope ^{14}C has a half-life of 5,715 years. A piece of ancient charcoal contains only 73% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal?

$t =$ ______________________

Remark: There will be 20 questions regarding the materials up to Exam 3 (including Exam 3) in the Final Exam. Look also at the "NOTE" for the subject matter 15 about computing the total distance.

5. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formulas for sine and cosine

$$\begin{cases} \sin(2x) = 2\sin(x)\cos(x) \\ \cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1 \end{cases}$$

and $\sin^2(x) + \cos^2(x) = 1$, etc.).

Example Problems

(5.1) How many solutions are there on the interval $[0, 2\pi]$ for the equation $\sqrt{3}\sin(x) = \sin(2x)$?

$x =$ ______________________

(5.2) Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\cos(2x) - \sin(2x) = 0$.

$x =$ ______________________________

(5.3) Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $2\sin^2(x) = \cos(2x)$.

$x =$ ______________________________

(5.4) Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\cos(2x) - \sin(x) = 0$.

$x =$ ______________________

(5.5) Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\sin^2(x) + \sin(x) = \cos^2(x)$.

$x =$ ______________________

(5.6) Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\sin(2x) - \cos(x) = 0$.

$x =$ ______________________________

6. You are supposed to know how to compute the derivative of the function of the form $y = f(x)^{g(x)}$.

(6.1) Find the derivative of the following functions:

(6.1.1) $y = (\sqrt{x})^{\tan(x)}$

$y' =$ ______________________________

(6.1.2) $y = (\sqrt[4]{x^3})^{\tan(x)}$

$y' =$ ____________________

(6.1.3) $y = x^{\cos(x)}$

$y' =$ ____________________

(6.1.4) $y = x^{\ln(x^2)}$

$y' =$ ____________________

(6.1.5) $y = x^{\ln(2x^3)}$

$y' =$ ____________________

$y' =$

(6.1.6) $y = (\ln(x))^{\sin(x)}$

$y' =$ ______________________________

7. **TWO "Related Rates" problems will be given in the Final Exam.**

(7.1) (Street Light) A person of 6 feet in height is walking towards a streetlight along a straight path at a rate of 3 feet per second. The height of the streetlight is 24 feet above the ground. What is the rate in feet/second at which the person's shadow is shortening?

WARNING: We are asking for the rate of change for the length of his shadow, NOT for the rate of change for the distance from the tip of the shadow to the bottom of the streetlight.

Answer:______________________________

(7.2) (Sand Pile) Sand is being dumped from a conveyor belt at the rate of 30 $\mathrm{ft}^3/\mathrm{min}$ and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

(HINT: The volume V of a circular cone with radius r for the bottom circle and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.)

(7.3) (Sand Pile) Sand is being dumped from a conveyor belt at the rate of $12\pi\text{ft}^3/\text{min}$ and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 6 ft high?

(HINT: The volume V of a circular cone with radius r for the bottom circle and height h is given by $V = \frac{1}{3}\pi r^2 h$.)

Answer:______________________________

(7.4) (Water tank with leak) A leaky water tank has the shape of an inverted cone. The tank at a rate of 10 m^3/min. At the moment when the water in the tank is 12 m deep, the water level is increasing at $\frac{1}{3}$ m/min. Assume that the tank leaks at a constant rate. What is the rate of leakage from the tank?

Answer:______________________________

(7.5) A paper conical cup has a radius 8 cm and height 16 cm. Suppose water is poured into the cup at a rate of 8 cm^3/s. How fast is the water level rising in cm/s when the water is 8 cm deep?

(HINT: The volume V of a reversed circular cone with radius r for the top circle and height h is given by $V = \frac{1}{3}\pi r^2 h$.)

Answer:______________________________

(7.6) (Ladder: speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Answer: ____________________

(7.7) (Ladder: angle) A 15-foot ladder is leaning against a vertical wall and its bottom is being pushed toward the wall at the rate of 3 ft/sec. At what rate is the acute angle the ladder makes with the ground is $\pi/4$?

Answer:____________________________________

(7.8) (Distance) Two people start from the same point at the same time. One walks north at 2 mi/h and the other walks west at 4 mi/h. How fast is the distance between them changing after 30 minutes?

Answer:______________________________

(7.9) (Water drip) Water is falling on the surface, wetting a circulaar area that is expanding at a rate of 9 mm^2/sec. How fast is the radius of the west area expanding when the radius is 135 mm?

Answer:____________________

(7.10) (Baseball) A mini-baseball diamond is a square ABCD with side 9 m. A batter hits the ball at A and runs towards first base B with a speed of 2 m/sec. At what rate is his distance from third base D increasing when he is two-thirds of the way to first base?

Answer:____________________

(7.11) (Area of a triangle) Suppose that the base of a triangle increases at the rate of 3 cm/s and the height increases at the rate of 4 cm/s. What is the rate of increase of the triangle when the base is 5 cm and the height is 6 cm?

Answer:____

(7.12) (Lighthouse) A lighthouse is located on a small island 4 km away from the nearest point P on a straight shoreline and its light makes two revolutions per minute. How fast is the beam of light moving along the shoreline when it is 3 km from P?

Answer:______________________________

8. You are supposed to understand the idea of the linear approximation $f(x) \approx L(x) = f(a) + f'(a)(x-a)$ of a function $f(x)$ at $x = a$ and apply it to approximate the value of a function at a given point.

Example Problems

(8.1) Find the formula for the linear approximation to $f(x) = \sqrt{x}$ at $a = 9$ and use it to approximate $\sqrt{9.3}$.

$\sqrt{9.3} \approx$ ______________________

(8.2) Find the formula for the linear approximation to $f(x) = \sqrt[3]{x}$ at $a = 27$ and use it to approximate $\sqrt[3]{24}$.

$\sqrt[3]{24} \approx$ ______________________

(8.3) What is the result of approximating $\sqrt[4]{16.32}$ using the linear approximation of $f(x) = \sqrt[4]{16+x}$ centered at $a = 0$?

$\sqrt[4]{16.32} \approx$ ____________________

(8.4) Use the linear approximation of the function $f(x) = \ln(1 + x)$ at $a = 0$ to find the estimate of $\ln(1.01)$.

$\ln(1.01) \approx$ ____________________

(8.5) Use the linear approximation of the function $f(x) = \ln(1 + 2x)$ at $a = 0$ to find the estimate of $\ln(1.01)$.

$\ln(1.01) \approx$ ____________________

(8.6) Use the linear approximation of the function $f(x) = e^{2x}$ at $a = 0$ to find the estimate of $e^{1.01}$.

$e^{1.01} \approx$ ____________________

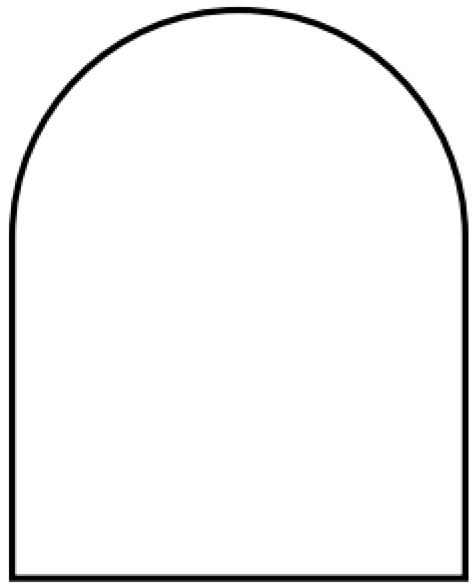
(8.7) Suppose $y = f(x)$ is a differentiable function (defined everywhere) that satisfies $f(2) = 3$ and $f'(2) = -4$. Find the estimate of $f(1.97)$ using the linear approximation of f at $a = 2$.

$f(1.97) \approx$ ____________________________

9. **TWO Optimization Problems will be given in the Final Exam.**

Example Problems

(9.1) A Norman window is a window in shape of a rectangle surmounted by a semicircle whose diameter equals the base of the rectangle. What is the largest possible area of a Norman window with perimeter 8 m?



(9.2) What is the smallest possible area of a triangle formed by the coordinate axes and a line tangent to the ellipse

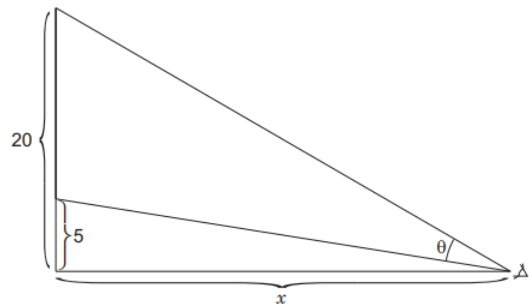
$$\frac{x^2}{7^2} + \frac{y^2}{3^2} = 1$$

in the first quadrant?

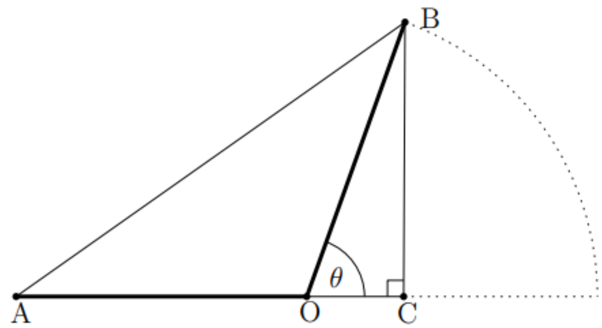
(9.3) What is the largest area of a right triangle whose hypotenuse has length 7?

(9.4) What is the maximum area of a rectangle whose base is on the x-axis, having the remaining two vertices on the graph of $y = 9 - x^2$ and lying above the x-axis?

(9.5) An auditorium with a flat floor has a large screen on one wall. The lower edge of the screen is 5 feet above eye level and the upper edge of the screen is 20 feet above eye level. How far from the screen should you stand to maximize your viewing angle?



(9.6) Two wooden bars of equal length $AO = BO = 1$ ft are connected by a hinge at point O so that one can rotate the bar BO around as shown in the picture below. Find the maximum area of the triangle ΔABC when $0 < \theta < \pi$.



(9.7) A rectangular box has to be made with the width being twice as long as the length (with a bottom but WITHOUT a top). If the surface area of the box is 400 cm^2, what is the height of the box with the largest volume?

(9.8) What is the maximum area of a rectangle whose base is on the x-axis, having the remaining two vertices on the graph of $y = \dfrac{4}{4+x^2}$?

10. You are supposed to understand the notions of continuity and differentiability, and their difference.

Example Problems

(10.1) Determine whether the following function is continuous/differentiable at the given point.

(10.1.1) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$ at $x = 0$.

Continuous at $x = 0$?________________

Differentiable at $x = 0$?________________

(10.1.2) $g(x) = \begin{cases} 4x & \text{if } x < 1 \\ x^3 + 3 & \text{if } x \geq 1 \end{cases}$ at $x = 1$.

Continuous at $x = 1$?________________

Differentiable at $x = 1$?________________

(10.1.3) $g(x) = \begin{cases} 5x & \text{if } x < 1 \\ x^3 + 4 & \text{if } x \geq 1 \end{cases}$ at $x = 1$.

Continuous at $x = 1$?____________________

Differentiable at $x = 1$?____________________

(10.1.4) $h(x) = |x - 2|(x - 5)|x - 5|$ at $x = 2$ and at $x = 5$

Continuous at $x = 2$?____________________

Differentiable at $x = 2$?____________________

Continuous at $x = 5$?____________________

Differentiable at $x = 5$?____________________

(10.2) Find the real numbers a and b so that the following function becomes continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} a & \text{if } x \leq 1 \\ \dfrac{12(\sqrt{x^2+8}-b)}{x^2-1} & \text{if } x > 1. \end{cases}$$

$a =$ ______________________

$b =$ ______________________

(10.3) Find the real numbers a and b so that the following function becomes continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} a & \text{if } x \leq 3 \\ \dfrac{|-5x + 5b|}{x - 3} & \text{if } x > 3. \end{cases}$$

$a =$ ______________________________

$b =$ ______________________________

(10.4) Find the real numbers a and b so that the following function becomes continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} 2x - a & \text{if } x \leq 0 \\ \dfrac{(\sqrt{x^2+4} - b)}{x^3 + x^2} & \text{if } x > 0. \end{cases}$$

$a =$ ____________________

$b =$ ____________________

(10.5) Find the real numbers a and b so that the following function becomes continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} x - b + 1 & \text{if } x \leq 4 \\ \dfrac{ax - b}{x - 4} & \text{if } x > 4. \end{cases}$$

$a =$ ______________________________

$b =$ ______________________________

11. Given the following function with the prescribed domain, find the formula, domain and range of its inverse function.

(11.1) Given the following function with the prescribed domain, find the formula, domain and range of its inverse function.

(11.1.1) $f(x) = \dfrac{1-2x}{x-6}$ defined over $(6, \infty)$

$f^{-1}(x) =$ ____________________

Range of $f^{-1}(x)$: ____________________

(11.1.2) $f(x) = \dfrac{-3x+2}{x-1}$ defined over $(-\infty, 1)$

$f^{-1}(x) =$ ____________________

Range of $f^{-1}(x)$: ____________________

(11.1.3) $f(x) = \dfrac{6x-1}{2x+1}$ defined over $(1, \infty)$

$f^{-1}(x) =$ ____________________

Range of $f^{-1}(x)$: ____________________

(11.1.4) $f(x) = \dfrac{e^x}{2e^x + 1}$ defined over $(-\infty, \infty)$

$f^{-1}(x) =$ ______________________

Range of $f^{-1}(x)$: ______________________

(11.1.5) $f(x) = 1 - \sqrt{x+1}$ defined over $[-1, \infty)$

$f^{-1}(x) =$ ____________________

Range of $f^{-1}(x)$: ____________________

12. You are supposed to understand the method of Implicit Differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems

(12.1) Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the equation $f(x) + x^2(f(x))^3 = 10$ and the condition $f(1) = 2$. Find $f'(1)$.

$f'(1) =$ ____________________

(12.2) Find the equation of the tangent line to the curve given by the equation $x^2 + y^2 + 6x + 3y - 5 = 0$ at the point $(1, -1)$.

$y =$ ____________________

(12.3) Find the slope of the tangent to the curve given by the equation $x^2 \sin(2y) = y^2 \cos(2x)$ at a point $(\pi/4, \pi/2)$.

$$\left.\frac{dy}{dx}\right|_{(\pi/4,\pi/2)} = \rule{5cm}{0.4pt}$$

(12.4) Find the slope of the tangent to the curve given by the equation $2xy + \pi \sin(xy) = 2\pi$ at a point $(\pi/2, 1)$.

$$\left.\frac{dy}{dx}\right|_{(\pi/2,1)} = \rule{5cm}{0.4pt}$$

(12.5) Find $\dfrac{dy}{dx}$ given $e^{x/y} = 7x - y$

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

(12.6) Find $\dfrac{dy}{dx}$ given $e^{xy} = 7x - y$

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

(12.7) Find $\dfrac{dy}{dx}$ given $\ln\left(\dfrac{x}{y}\right) = \pi(7x - y)$

$$\frac{dy}{dx} = \underline{\hspace{6cm}}$$

13. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are also supposed to know the relation between the derivative of the original function and the derivative of its inverse (when it exists).

    Example problems

    (13.1) Compute the derivative of the following function:

    (13.1.1) $y = \sin(\sin(\sin(x)))$

$$y' = \underline{\hspace{6cm}}$$

(13.1.2) $y = \left(\dfrac{t-2}{2t+1}\right)^9$

$y' =$ ____________________________

(13.1.3) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

$y' =$ ____________________________

(13.1.4) $y = \ln|\sec(3\theta) + \tan(3\theta)|$

$y' =$ ____________________

(13.2) Suppose that $F(x) = f^{-1}(\{g(x)\}^2)$ and that the functions f (which is one-to-one, and hence has its inverse) and g satisfy the following conditions:

$$\begin{cases} f(2) = 9, \quad f(9) = 5, \\ f'(1) = 4, \quad f'(2) = 3, \quad f'(3) = -2 \\ g(1) = 3, \quad g'(1) = 2 \end{cases}$$

Find $F'(1)$.

$F'(1) =$ ____________________

(13.3) Suppose that g is a function with $g'(1/2) = 5$. Let f be the function defined by

$$f(x) = g(2\cos^2(x))$$

Compute $f'(\pi/3)$.

$f'(\pi/3) =$ ______________________

(13.4) Suppose that g is a function with $g'(1/2) = 5$. Let f be the function defined by

$$f(x) = g(2\sin^2(x))$$

Compute $f'(\pi/6)$.

$f'(\pi/6) =$ ______________________

(13.5) Suppose that

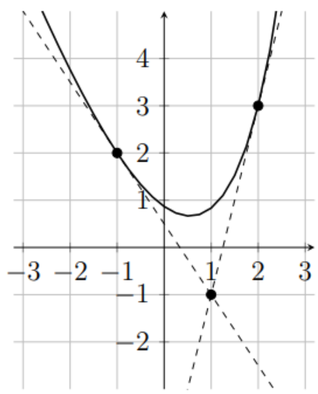
$$F(x) = \frac{1}{g(f^{-1}(x))}$$

and that the functions f (which is one-to-one, and hence has its inverse) and g satisfy the following conditions.

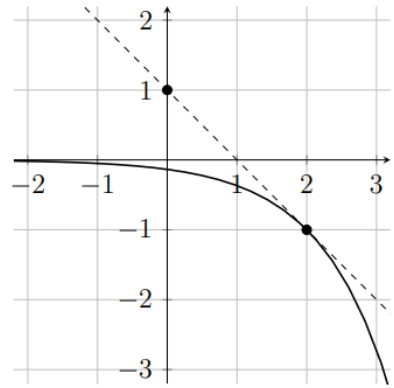
$$\begin{cases} f(2) = 4 & f'(2) = 8 \\ g(2) = 3 & g'(2) = 2 \\ g(5) = 2 & g'(5) = 7 \end{cases}$$

$F'(4) =$ ____________________

(13.6) Let $f(x)$ and $g(x)$ be two differentiable functions with the following graphs.



Graph of $f(x)$



Graph of $g(x)$

The dashed lines in the figures are tangent to the graphs at the indicated points. What is the derivative of $\{g(f(1-x))\}^2$ at $x = 2$?

$\{g(f(1-x))\}^2$ at $x = 2$ is ________________

14. Given the position function of a particle, you are supposed to be able to compute its velocity, acceleration, understanding its physical meaning. You should be able to determine when a particle is speeding up or down, whether it is accelerating or decelerating. You are supposed to be able to compute the "total" distance traveled during the given period.

Example Problems

(14.1) The position of a particle is given by the function $s = f(t) = t^3 - 6t^2 + 9t$. Find the total distance traveled during the first 6 seconds.

Total distance for the first 6 seconds: __________

(14.2) The velocity function, which is the 1st derivative of a position function $f(t)$ is given by the following formula $v = f'(t) = 6(t-1)(t-3)$. Can you determine the total distance traveled during the first 5 seconds?

Total distance for the first 5 seconds: ______________

(14.3) A rock is thrown upward so that its height (in ft) after t seconds is given by $h(t) = 48t - 16t^2$. What is the velocity of the rock when its height is 32 ft on its way up?

Velocity when $h(t) = 32$: ______________

(14.4) A rocket is launched upward so that its height (in ft) after t seconds is given by

$$h(t) = 32t - 16t^2$$

What is the velocity of the rocket when its height is 12 ft on its way up?

Velocity when $h(t) = 12$: ____________

(14.5) The position of a toy rocket above ground is given by $y(t) = 20(t - t^2)e^t$ meters after t seconds starting at $t = 0$. What is the speed at which the rocket hit the ground when it comes down to the ground?

Speed of rocket when it comes down: ____________

15. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function. You are also supposed to find the inflection points by analyzing the behavior of the 2nd derivative.

Example Problems

(15.1) The first derivative of a function f is given by

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5)$$

Find the values of x for which the function f takes

(a) local minimum, and (b) local maximum.

Local Min: ____________________

Local Max: ____________________

(15.2) The first derivative of a function f is given by

$$f'(x) = (x+2)^2(x+3)^3(x-1)^2(x-2)(x-4)^5$$

Find the values of x for which the function f takes

(a) local minimum, and (b) local maximum.

Local Min: ____________________

Local Max: ____________________

(15.3) Consider the function $f(x) = x^8(x-4)^7$.

(a) Find the critical numbers of the function f.

(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?

(c) What does the First Derivative Test tell you that the Second Derivative test does not?

Critical Points:____________________

Second Derivative Test says:____________________

First Derivative Test says:____________________

(15.4) How many inflection points does the graph of the function $y = f(x) = x^5 - 5x^4 + 25x$ have?

Answer:____________________ # of inflection points:

(15.5) We have a function whose first derivative is given by the formula $f'(x) = (x-1)^5(x+1)^5$. Find the x-coordinates of the local extrema and the inflection points of the function.

$x =$ ______________________________

(15.6) We have a function whose first derivative is given by the formula $f'(x) = (x-2)^3(x+2)^3$. Find the x-coordinates of the local extrema and the inflection points of the function.

$x =$ ______________________________

16. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down). You are also supposed to be able to determine the horizontal/vertical/slant asymptotes of the graph.

Example Problems

(16.1) Draw the graph of the following function:

(16.1.1) $y = f(x) = \dfrac{x^3}{x^2 - 16}$

(16.1.2) $y = f(x) = \dfrac{x^2}{x^2 - 16}$

(16.1.3) $y = f(x) = \dfrac{x}{x^2 - 16}$

(16.1.4) $y = f(x) = \dfrac{1}{x^2 - 16}$

(16.1.5) $y = f(x) = \dfrac{x}{x^2 + 16}$

(16.1.6) $y = f(x) = \dfrac{x}{1 - x^2}$

(16.1.7) $y = f(x) = \dfrac{x}{4x^2 - 25}$

(16.1.8) $y = f(x) = \ln|x^2 - 10x + 24|$

(16.2) Find the equations of the horizontal, vertical, and slant asymptotes of the function

$$f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}$$

VA:________________________________

HA:________________________________

SA:________________________________

17. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$, and the values on the critical values(s). You should know what the definition of a critical value is.

Example Problems

(17.1) Find the absolute maximum and absolute minimum values of the function f on the given interval.

(17.1.1) $f(x) = x^{2/3}$ on $[-1, 8]$

Absolute Min:____________________________

Absolute Max:____________________________

(17.1.2) $f(x) = \dfrac{\cos(x)}{2 - \sin(x)}$ on $[-\pi/2, \pi/2]$

Absolute Min:____________________________

Absolute Max:____________________________

(17.1.3) $f(x) = 2x^3 + 3x^2 - 12x$ on $[-1, 2]$

Absolute Min:____________________

Absolute Max:____________________

(17.1.4) $f(t) = 2\cos(t) + \sin(2t)$ on $[0, 2\pi]$

Absolute Min:____________________

Absolute Max:____________________

(17.1.5) $f(x) = 4\cos^3(x) + 21\cos^2(x) - 24\cos(x)$ on $[-\pi/2, \pi/2]$.

Absolute Min:____________________________

Absolute Max:____________________________

(17.1.6) $f(x) = e^{-x} \cdot \sin(x)$ on $[0, 3\pi/2]$

Absolute Min:____________________________

Absolute Max:____________________________

(17.1.7) $f(x) = x - 3\ln(x)$ on $[1, 4]$

Absolute Min:________________________________

Absolute Max:________________________________

18. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply this corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise: If $f'(x) = 0$ for all values of $x \in (a, b)$, then a continuous function f on the closed interval $[a, b]$ is actually a constant.

Example Problems

(18.1) Consider the function $f(x) = x^4 - 2x^2 + 5x + 3$ over the interval $[-2, 2]$. How many values of $c \in (-2, 2)$ satisfy the statement of the Mean Value Theorem: $f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}$?

of $c \in (-2, 2)$: ______________________________

(18.2) Consider the function $f(x) = x(x^4 - 5)$ over the interval $[a, b] = [0, 2]$. What value(s) of $c \in (a, b)$ satisfies the statement of the Mean Value Theorem: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

\# of $c \in (0, 2)$: ____________________

(18.3) Consider the function $f(x) = x^{2/3}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem to hold? Do we have any value $c \in (-1, 1)$ such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)}$?

Does MVT hold? ____________________

\# of $c \in (-1, 1)$ ____________________

(18.4) Determine the exact value of $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(7)$.

$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(7) =$ ____________________

(18.5) Consider the function

$$f(x) = \tan^{-1}(x) - \tan^{-1}\left(-\frac{1}{x}\right)$$

(i) What can you say about $f'(x)$??

(ii) What is the conclusion derived from (i)?

(iii) Determine the value of $f(7) = \tan^{-1}(7) - \tan^{-1}\left(-\frac{1}{7}\right)$

(iv) Determine the value of $f(-7) = \tan^{-1}(-7) - \tan^{-1}\left(\frac{1}{7}\right)$

(v) Do we have $f(7) = f(-7)$? If not, does it contradict (i) and (ii)?

19. You are supposed to be able to compute the (right/left hand side) limit, understanding its proper meaning, and using the Squeeze Theorem. You are also supposed to be able to determine the exact value of the limit who has an indeterminate form (e.g., $0/0, \pm\infty/\pm\infty, \infty - \infty$ using some proper technique.

(19.1) Compute the following limits:

(19.1.1) $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 7x + 12}$

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 7x + 12} = \rule{5cm}{0.4pt}$$

(19.1.2) $\lim_{x \to 5^-} \frac{x^2 - 4x - 5}{|x - 5|}$

$$\lim_{x \to 5^-} \frac{x^2 - 4x - 5}{|x - 5|} = \rule{5cm}{0.4pt}$$

(19.1.3) $\lim_{x \to (\pi/2)^+} e^{\tan(x)}$

$\lim_{x \to (\pi/2)^+} e^{\tan(x)} =$ ____________________

(19.1.4) $\lim_{x \to 0} \left(\frac{\sqrt{5x^2+9}-3}{x^2} \right)$

$\lim_{x \to 0} \left(\frac{\sqrt{5x^2+9}-3}{x^2} \right) =$ ____________________

(19.1.5) $\lim_{x\to 0}\left(\frac{5}{x^2-x}+\frac{5}{x}\right)$

$$\lim_{x\to 0}\left(\frac{5}{x^2-x}+\frac{5}{x}\right) = \underline{\hspace{4cm}}$$

(19.1.6) $\lim_{x\to\infty}(x^2-\sqrt{x^4+16x^2})$

$$\lim_{x\to\infty}(x^2-\sqrt{x^4+16x^2}) = \underline{\hspace{4cm}}$$

(19.1.7) $\lim_{x\to\infty}(x^2 - \sqrt{x^4 + 9x^2})$

$\lim_{x\to\infty}(x^2 - \sqrt{x^4 + 9x^2}) =$ ______________________

(19.1.8) $\lim_{x\to\infty}(\sin(x)) \cdot \tan\left(\frac{3}{x}\right)$

$\lim_{x\to\infty}(\sin(x)) \cdot \tan\left(\frac{3}{x}\right) =$ ______________________

(19.1.9) $\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2}$

$\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2} =$ ____________________

(19.1.10) $\lim_{x \to \infty} \frac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$

$\lim_{x \to \infty} \frac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}} =$ ____________________

(19.1.11) $\lim_{x\to-\infty} \dfrac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$

$$\lim_{x\to-\infty} \frac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}} = \underline{\qquad\qquad}$$

20. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form $0/0$, $\pm\infty, \pm\infty$

Example Problems

(20.1) Compute the limits:

(20.1.1) $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \underline{\qquad\qquad}$$

(20.1.2) $\lim_{x \to \pi} \frac{1 + \cos(x)}{(x - \pi)^2}$

$$\lim_{x \to \pi} \frac{1 + \cos(x)}{(x - \pi)^2} = \_\_\_\_\_\_\_\_\_\_\_\_$$

(20.1.3) $\lim_{x \to 0} \frac{\sin(x^2)}{\arctan(\cos(x) - 1)}$

$$\lim_{x \to 0} \frac{\sin(x^2)}{\arctan(\cos(x) - 1)} = \_\_\_\_\_\_\_\_\_\_\_\_$$

(20.1.4) $\lim_{x\to 0} \frac{7^x - 6^x}{3^x - 2^x}$

$\lim_{x\to 0} \frac{7^x - 6^x}{3^x - 2^x} =$ ____________________

(20.1.5) $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} =$ ____________________

(20.1.6) $\lim_{x \to 0} \frac{\tan(x) - x}{x^3}$

$$\lim_{x \to 0} \frac{\tan(x) - x}{x^3} = \underline{\hspace{4cm}}$$

(20.1.7) $\lim_{x \to 0^+} \frac{\sin(x)}{1 - x^2}$

$$\lim_{x \to 0^+} \frac{\sin(x)}{1 - x^2} = \underline{\hspace{4cm}}$$

(20.1.8) $\lim_{x \to 0} \frac{\ln(\cos(2x))}{x^2}$

$\lim_{x \to 0} \frac{\ln(\cos(2x))}{x^2} =$ ______________________

(20.1.9) $\lim_{x \to 0} \frac{2x - \tan(2x)}{4x - \sin(4x)}$

$\lim_{x \to 0} \frac{2x - \tan(2x)}{4x - \sin(4x)} =$ ______________________

21. You are suppose to know how to compute the limits of the indeterminate form $\pm\infty \times 0, \infty - \infty$.

Example Problems

(21.1) Compute the following limits.

(21.1.1) $\lim_{x\to 0^+} \sin(x) \cdot \ln(2x)$

$\lim_{x\to 0^+} \sin(x) \cdot \ln(2x) =$ ______________________

(21.1.2) $\lim_{x\to\infty} \left[x \cdot \ln\left(\frac{2\tan^{-1}(x)}{\pi} \right) \right]$

$\lim_{x\to\infty} \left[x \cdot \ln\left(\frac{2\tan^{-1}(x)}{\pi} \right) \right] =$ ________________

(21.1.3) $\lim x \to (\pi/2)^{-1}[(2x - \pi) \cdot \tan(x)]$

$\lim x \to (\pi/2)^{-1}[(2x - \pi) \cdot \tan(x)] =$ ____________

(21.1.4) $\lim_{x\to\infty} (\sqrt{x^2 + 3x + 5} - x)$

$\lim_{x\to\infty} (\sqrt{x^2 + 3x + 5} - x) =$ ____________

(21.1.5) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) =$ ____________________

(21.1.6) $\lim_{x\to\infty} 2x\cdot\tan\left(\frac{1}{3x}\right)$

$\lim_{x\to\infty} 2x\cdot\tan\left(\frac{1}{3x}\right) =$ ____________________

22. You are supposed to be able to compute the limits $\lim_{x\to a}[f(x)]^{g(x)}$ formally of the form $0^0, \infty^0, 1^\infty$

Example Problems

(22.1) Compute the following limits.

(22.1.1) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x}$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x} = \rule{4cm}{0.4pt}$$

(22.1.2) $\lim_{x\to\infty}\left(\frac{2x+1}{3x-1}\right)^{3x+1}$

$$\lim_{x\to\infty}\left(\frac{2x+1}{3x-1}\right)^{3x+1} = \rule{4cm}{0.4pt}$$

(22.1.3) $\lim_{x\to\infty}(2x+e^{5x})^{1/x}$

$\lim_{x\to\infty}(2x+e^{5x})^{1/x} =$ ______________________

(22.1.4) $\lim_{x\to 0^+}[\tan(5x)]^{\sin(x)}$

$\lim_{x\to 0^+}[\tan(5x)]^{\sin(x)} =$ ______________________

(22.1.5) $\lim_{x\to 0}(1+3\sin(x))^{7/x}$

$\lim_{x\to 0}(1+3\sin(x))^{7/x} =$ ____________________

(22.1.6) $\lim_{x\to 4^+}\left(\frac{x}{4}\right)^{\frac{12}{x-4}}$

$\lim_{x\to 3^+}\left(\frac{x}{3}\right)^{\frac{6}{x-3}} =$ ____________________

(22.1.7) $\lim_{x \to 0^+} (1 + 3x)^{\cot(x)}$

$\lim_{x \to 0^+} (1 + 3x)^{\cot(x)} =$ ____________________