

★ Section 6.2 - Part 2

Linear & Almost Linear Systems

Warm up: Find the Jacobian $\underline{J}$ at $(0,0)$ of the nonlinear system:

$$x' = 4y + 7y^2 - 3x^5y = F(x,y)$$

$$y' = x - 17y^5 + 6x^2y = G(x,y)$$

$$\underline{J} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \bigg|_{(0,0)}$$

$$= \begin{bmatrix} -15x^4y & 4 + 14y - 3x^5 \\ 1 + 12xy & -5(17)y^4 + 6x^2 \end{bmatrix} \bigg|_{(0,0)}$$

$$\underline{J} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

I. Evaluating Almost Linear Systems:

An autonomous system:

$$x' = F(x,y)$$

$$y' = G(x,y)$$

has a critical point (x_*, y_*) when

$$F(x_*, y_*) = 0 = G(x_*, y_*)$$

Use Taylor series expansion to get the almost linear system:

$$(*) \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_*, y_*) & F_y(x_*, y_*) \\ G_x(x_*, y_*) & G_y(x_*, y_*) \end{bmatrix}}_{\underline{J} - \text{Jacobian matrix}} \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix} + \begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}$$

The associated linear system is:

$$(\square) \quad \underline{x}' = \underline{J} \underline{x} \quad \text{where} \quad \underline{x} = \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix}$$

Thm: The almost linear system (*) will have the same type of critical point and the

Thm: The almost linear system (*) will have the same type of critical point and the same stability as the linear system (0)

UNLESS: (a) $\lambda_1 = \lambda_2$
 type: node or spiral point
 stable if $\lambda_1 = \lambda_2 < 0$
 unstable if $\lambda_1 = \lambda_2 > 0$

(b) $\lambda = \pm bi$
 type: center or spiral
 may be either unstable, stable, or asymptotically stable.

Ex:
$$\begin{aligned} x' &= 4x + 7y - (x^2 + 3y^2) && r(x,y) \\ y' &= x - 2y - 13xy && s(x,y) \end{aligned}$$

Taylor Series

$$F(x,y) = F(0,0) + F_x(0,0)x + F_y(0,0)y + r(x,y)$$

1. Claim: (0,0) is a critical point

2. Calculate the Jacobian

$$\begin{aligned} \underline{J} &= \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \bigg|_{(0,0)} \\ &= \begin{bmatrix} 4-2x & 7+6y \\ 1-13y & -2-13x \end{bmatrix} \bigg|_{(0,0)} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

Associated Linear system:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underline{J} \begin{bmatrix} x \\ y \end{bmatrix}$$

keep all the linear terms from autonomous system

$$\begin{aligned} x' &= 4x - 7y \\ y' &= x - 2y \end{aligned}$$

3. Evaluate the phase portrait of the linear system:

$$\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$$

Char. eqn

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 - 2\lambda - 15 = 0$$

$$(\lambda - 5)(\lambda + 3) = 0$$

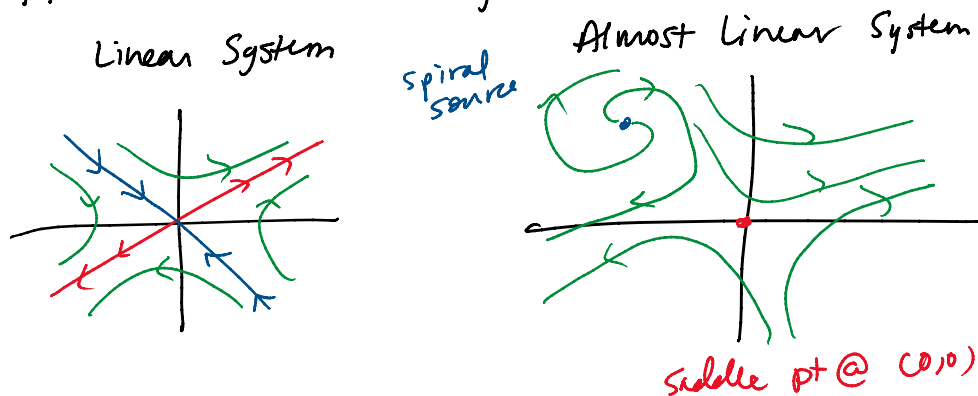
$$\lambda = -3, +5$$

$$T = \text{trace}(\underline{J}) = 4 - 2 = 2$$

$$D = \det(\underline{J}) = -8 - 7 = -15$$

So (0,0) is a saddle point and unstable for the linear system.

By the Thm, $(0,0)$ is also a saddle and unstable for the almost linear system.



Ex: Find the critical point (x_*, y_*) and apply the Thm to classify its type and Stability.

$$\begin{aligned}x' &= x - 2y \\y' &= 3x - 4y - 2\end{aligned}$$

1. Find the C.P.

$$x - 2y = 0$$

$$x = 2y$$

$$x = 2$$

$$3x - 4y - 2 = 0$$

$$3(2y) - 4y = 2$$

$$2y = 2$$

$$y = 1$$

$$(x_*, y_*) = (2, 1)$$

2. Linearize $\rightarrow$ Find Jacobian

$$\underline{J} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \bigg|_{(2,1)} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \bigg|_{(2,1)} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Associated Linear System

$$\underline{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \underline{x}$$

3. Find the eigenvalues of $\underline{J}$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$T = \text{tr}(\underline{J}) = 1 - 4 = -3$$

$$D = \det(\underline{J}) = -4 - 6 = -2$$

$$(\lambda+2)(\lambda+1)=0$$

$$\lambda = -1, -2$$

4. Classify the c.p.

e.v. are real, neg $\rightarrow$ improper nodal sink
and asymptotically stable

5. By the Thm, c.p. $(2,1)$ is a improper nodal sink and asymptotically stable

Thm: If λ_1 and λ_2 are eigenvalues of $\underline{x}' = \underline{A}\underline{x}$

1. $\text{Re}(\lambda) < 0$ then asymptotically stable
2. $\text{Re}(\lambda) = 0$ then stable
3. Otherwise unstable

Ex: The following system has a c.p. at $(0,0)$.
Classify its type and stability using Thm 2.

$$x' = -6x + 4y - x^2 + 3y^2$$

$$y' = -4x + 2y - 20xy$$

1. Linearize $\rightarrow$ Find the Jacobian

$$\underline{J} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \bigg|_{(0,0)} = \begin{bmatrix} -6-2x & 4+6y \\ -4-20y & 2-20x \end{bmatrix} \bigg|_{(0,0)}$$

$$= \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix}$$

Associated Linear System:

$$\underline{x}' = \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} \underline{x}$$

2. Evaluate linear system $\rightarrow$ calculate λ

$$\lambda^2 - T\lambda + D = 0$$

$$T = \text{tr}(\underline{J}) = -6 + 2 = -4$$

$$D = \det(\underline{J}) = -12 + 16 = 4$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0 \quad \text{multiplicity}$$

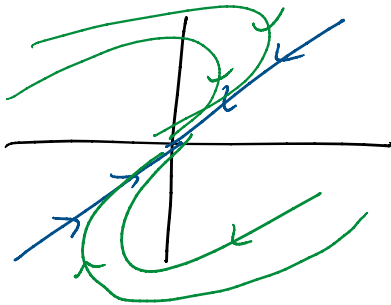
$$\lambda = -2 \quad \text{with } k=2$$

check that λ is defective

3. The linear system is an improper nodal sink
 b/c $\lambda = -2 < 0 \rightarrow$ asymptotically stable

4. By Thm 2, because $\lambda_1 = \lambda_2$, all we can say is that $(0,0)$ is either a node or a spiral, but it still is asymptotically stable

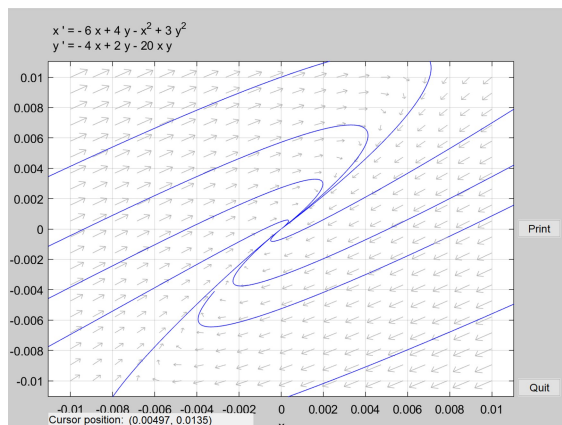
Linear System



zoom in on $(0,0)$
 improper nodal sink
 asymptotically stable

Almost Linear System

pplane 8 in MATLAB



Ex: Find and classify all the critical points

$$\begin{cases} x' = y^2 - 1 \\ y' = x^3 - y \end{cases}$$

1. Find the critical points

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

$$x^3 - y = 0$$

$$x^3 = y$$

$$x^3 = +1$$

$$x = +1$$

$$x^3 = -1$$

$$x = -1$$

Two critical points: $(1,1)$ and $(-1,-1)$

2. Linearize around each c.p.

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$$

$$@ (1,1) \quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix} \bigg|_{(1,1)} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$$

associated linear system: $\underline{x}' = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix} \underline{x}$

$$@ (-1,-1) \quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix} \bigg|_{(-1,-1)} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$$

associated linear system: $\underline{x}' = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix} \underline{x}$

3. Classify the c.p.

$$@ (1,1) \quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$$

Find the λ

$$T = -1 \quad D = -6$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda - 2)(\lambda + 3) = 0$$

$$\lambda = 2, -3$$

The c.p. (1,1) is
a saddle point
and unstable

$$@ (-1,-1) \quad \underline{\underline{J}} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$$

$$T = -1 \quad D = 6$$

$$\lambda^2 + \lambda + 6 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 6}}{2}$$

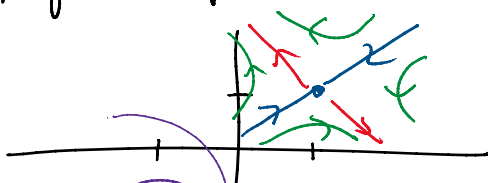
$$= \frac{-1 \pm \sqrt{1 - 24}}{2}$$

$$= \frac{-1 \pm i\sqrt{23}}{2}$$

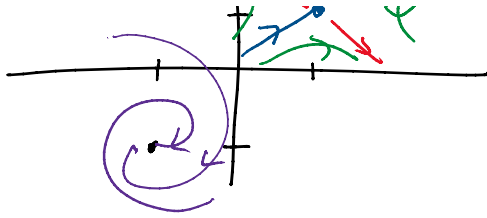
The c.p. (-1,-1) is
spiral sink

and since
 $\text{Re}(\lambda) = \frac{-1}{2} < 0$
asymptotically stable

Sketch phase portrait



see more like
this in
next lecture



next lecture