Last class, we stated the following Theorem

Thm 1: (Sturm-Liouville Eigenvalues)
Suppose $p(x), p'(x), q(x), r(x)$ are all continuous on $[a,b]$ and that
$p(x) > 0$ and $r(x) > 0$ at each point in $[a,b]$

Then the eigenvalues form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

with $\lim_{n\to\infty} \lambda_n = +\infty$

Furthermore, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ then all λ's are also nonnegative ($\lambda_n \geq 0$)

Def: A Sturm-Liouville Problem is called regular if
$p(x), p'(x), q(x), r(x)$ are continuous and
$p(x), r(x), q(x), \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Theorem 2: (Orthogonality of eigenfunctions)
Let the Sturm-Liouville Problem be regular. Let $y_i(x)$ and $y_j(x)$ be eigenfunctions associated with λ_i and λ_j ($\lambda_i \neq \lambda_j$)

Then $\int_a^b y_i(x)\, y_j(x)\, r(x)\,dx = 0$

Ex 1: Let
$$\begin{cases} y'' + \lambda y = 0 & 0 < x < \pi \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

Here $p(x) = 1, q(x) = 0, r(x) = 1$, $\alpha_1 = 1 \quad \alpha_2 = 0$ $\Big\} \Rightarrow$ We have a regular Sturm-Liouville Problem.

Here $p(x)=1, q(x)=0,$
$$\left.\begin{array}{ll} \alpha_1 = 1 & \alpha_2 = 0 \\ \beta_1 = 1 & \beta_2 = 0 \end{array}\right\} \Rightarrow \text{Sturm-Liouville Problem.}$$

Let's check that indeed this regular Sturm-Liouville Problem's eigenfunctions are orthogonal.

So $y''+\lambda y = 0 \Rightarrow y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

when $y(0)=0 \Rightarrow$
$$0 = A\cos(0) + B\sin(0)$$
$$0 = A$$

So $y(x) = B\sin(\sqrt{\lambda}x)$

when $y(\pi)=0 \Rightarrow 0 = B\sin(\sqrt{\lambda}\pi)$
$$0 = \sin(\sqrt{\lambda}\pi)$$
$$n\pi = \sqrt{\lambda}\pi$$
$$n = \sqrt{\lambda}$$
$$n^2 = \lambda \quad \text{for } n = 1, 2, \ldots$$

Note $B \neq 0$ b/c we get the trivial solution

Hence $y_n(x) = \sin(nx)$

Now let's check orthogonality: Assume $\lambda_i \neq \lambda_j$. So
$$y_i(x) = \sin(ix) \qquad y_j(x) = \sin(jx) \qquad r(x) = 1$$

We need to show $\int_0^\pi \sin(ix)\sin(jx)\cdot 1\,dx \stackrel{?}{=} 0$

We have used many times the idea that
$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \begin{cases} \pi/2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Since $\lambda_i \neq \lambda_j$, then $i \neq j$. So

Indeed $\int_0^\pi \sin(ix)\sin(jx)\,dx = 0$

Eigenfunction expansians

If $y_n(x)$ are eigenfunctions of a regular Strum-Liouville Problem, then we can write any function $f(x)$ as a linear combination of $y_n(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where c_n are constant coefficients.

A perfect example of this are our Fourier series with

$$y_n(x) = \sin(nx)$$

$$\underbrace{f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)}_{\text{Fourier Series}} \quad \text{so } c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

The question is how to do this (finding c_n) when our function isn't a fourier series. The key to finding such c_n is using orthogonality.

So $f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$

Multiply both sides by $y_m(x)r(x)$

$$f(x)y_m(x)r(x) = \sum_{n=1}^{\infty} c_n y_n(x) y_m(x) r(x)$$

Integrate both sides over $[a,b]$ w/ respect to x.

$$\int_a^b f(x)y_m(x)r(x)\,dx = \int_a^b \sum_{n=1}^{\infty} c_n y_n(x) y_m(x) r(x)\,dx$$

$$= \sum_{n=1}^{\infty} c_n \int_a^b y_n(x) y_m(x) r(x)\,dx$$

$$= \sum_{n=1}^{\infty} c_n \underbrace{\int_a^b y_n(x)\, y_m(x)\, r(x)\, dx}_{\text{Use orthogonality}}$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ ? & \text{if } n = m \end{cases}$$

$$= c_m \int_a^b [y_m(x)]^2 r(x)\, dx$$

So $c_m = \dfrac{\int_a^b f(x)\, y_m(x)\, r(x)\, dx}{\int_a^b [y_m(x)]^2 r(x)\, dx}$

Ex 2: Consider

$$\begin{cases} y'' + \lambda y = 0 & 0 < x < \pi/2 \\ y(0) = 0 \\ y'(\pi/2) = 0 \end{cases}$$

We can see this is a Sturm-Liouville Problem.

Remember $y'' + \lambda y = 0 \Rightarrow y(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$

When $y(0) = 0$: $0 = y(0) = A\cos(0) + B\sin(0)$

$$A = 0$$

So $y(x) = B\sin(\sqrt{\lambda}\, x)$

When $y'(\pi/2) = 0$: $y' = B\sqrt{\lambda}\cos(\sqrt{\lambda}\, x)$

$$0 = y'(\pi/2) = B\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}\,\pi}{2}\right)$$

So $\cancel{B = 0}$ or $0 = \cos\left(\frac{\sqrt{\lambda}\,\pi}{2}\right)$

b/c if $B=0$ then we have the trivial solution

$\frac{n\pi}{2} = \frac{\sqrt{\lambda}\,\pi}{2}$ when n is odd

$n = \sqrt{\lambda}$

$n^2 = \lambda_n$ when n is odd

Note equivalently we can say

$(2n-1)^2 = \lambda_n$ for $n = 1, 2, 3, \ldots$

$(2n-1) = \lambda_n$ for $n = 1, 2, \ldots$

So eigenvalues: $\lambda_n = (2n-1)^2$
eigenfunctions: $\sin((2n-1)x)$

$$\left.\begin{array}{l}\end{array}\right\} \Rightarrow f(x) = \sum_{m=1}^{\infty} c_m y_m(x) = \sum_{m=1}^{\infty} c_m \sin((2m-1)x)$$

Now we need to find c_m.

$$c_m = \frac{\int_a^b f(x)\, y_m(x)\, r(x)\, dx}{\int_a^b [y_m(x)]^2 r(x)\, dx}$$

$r(x) = 1$ from the ODE

$$= \frac{\int_0^{\pi/2} f(x) \sin((2m-1)x)\, dx}{\int_0^{\pi/2} [\sin((2m-1)x)]^2\, dx}$$

Let's calculate the denominator

$$\int_0^{\pi/2} [\sin((2m-1)x)]^2\, dx = \int_0^{\pi/2} \frac{1-\cos(2(2m-1)x)}{2}\, dx$$

$$= \frac{x}{2} + \frac{1}{2} \cdot \frac{\sin(2(2m-1)x)}{2(2m-1)} \Bigg]_0^{\pi/2}$$

$$= \frac{\pi/2}{2} + \frac{\sin(2(2m-1)\pi/2)}{2(2m-1)} - 0 - \cancel{\frac{1}{2} \frac{\sin(0)}{2(2m-1)}}^{\,0}$$

$$= \frac{\pi}{4} + \underbrace{\cancel{\frac{\sin((2m-1)\pi)}{2(2m-1)}}^{\,0}}_{\text{b/c } \sin(\#\pi) = 0}$$

where # is a integer and odd #s are integers.

$$c_m = \frac{4}{\pi} \int_0^{\pi/2} f(x) \sin((2m-1)x)\, dx$$

So $$f(x) = \sum_{m=1}^{\infty} \left(\frac{4}{\pi} \int_0^{\pi/2} f(x) \sin((2m-1)x)\, dx \right) \sin((2m-1)x)$$

<u>Ex 3:</u> Let's continue with Ex 2 and let $f(x) = 1$

Ex 3: Let's continue with Ex 2 and let $f(x)=1$

So $c_m = \frac{4}{\pi}\int_0^{\pi/2} \sin((2m-1)x)\,dx$

$$= \frac{4}{\pi}\cdot \frac{-\cos((2m-1)x)}{(2m-1)}\Bigg]_0^{\pi/2}$$

$$= \frac{4}{\pi}\left[\cancel{\frac{\cos\left(\frac{(2m-1)\pi}{2}\right)}{(2m-1)}}^{\to 0} + \frac{\cos(0)}{(2m-1)}\right]$$

b/c $\cos\left(\frac{\#\pi}{2}\right)=0$

where # is odd

$$= \frac{4}{\pi}\cdot\frac{1}{(2m-1)}$$

Hence $1 = f(x) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin((2m-1)x)$ is the eigenfunction expansion.

Thm 3: (Convergence of Eigenfunction Series)
Let $y_1, y_2, y_3, \ldots$ be the eigenfunctions of the Sturm-Liouville Problem on $[a,b]$. If $f(x)$ is piecewise smooth on $[a,b]$ then the eigenfunction series converges for $a<x<b$ to:

(i) the value $f(x)$ where f is continuous

(ii) the average value $\frac{1}{2}[f(x^+)+f(x^-)]$ at each point of discontinuity.