Section 7.1: Laplace Transforms and Inverse Transforms (Pt 1)

Recall the ODE for a mass on a spring.

$$mx'' + rx' + kx = F(t)$$

This ODE is a linear scalar ODE. Like in Ch. 5 and 6, we recalled that to solve ODE of this fashion we guessed the solutions to be of the type $x(t) = e^{rt}$ (or using the forcing term $F(t)$, like in Ch. 9, to inform an educated guess.

In this chapter, we will learn a new technique, Laplace Transforms, to solve more types of ODEs.

<u>Def:</u> Let f be a real-valued function of the real variable t, defined for $t>0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

E

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

This function F defined by the integral is called the <u>Laplace transform</u> of the function f.

<u>Ex 1:</u> Find the Laplace Transform of $f(t) = 1$, $t>0$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt = \lim_{R\to\infty} \int_0^R e^{-st}\,dt$$

$$= \lim_{R\to\infty} \left(\frac{e^{-st}}{-s}\right)\Big]_0^R$$

$$= \lim_{R\to\infty} \left(\frac{e^{-sR}}{-s} - \frac{e^{-s\cdot 0}}{-s}\right)$$

$$= \lim_{R\to\infty} \left(\underbrace{-\frac{e^{-sR}}{s}}_{\text{Converges to 0 when } s>0} + \frac{1}{s}\right)$$

when $s>0$

$$= \frac{1}{s} \text{ if } s>0$$

Hence $\mathcal{L}\{1\} = \frac{1}{s}$ if $s>0$

Note $\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$. We can see this via the next example.

Ex 2: Find the Laplace Transform of $f(t)=3$

$$\mathcal{L}\{3\} = \int_0^\infty 3e^{-st}dt = 3\int_0^\infty 1\cdot e^{-st}dt = 3\mathcal{L}\{1\} = 3\cdot\frac{1}{s} \text{ if } s>0$$

So $\mathcal{L}\{3\} = \frac{3}{s}$ if $s>0$.

Written HW: Find the Laplace Transform for $f(t)=25t$ for $t>0$

Ex 3: Find the Laplace Transform of $f(t)=e^{at}$ for $t>0$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \lim_{R\to\infty}\int_0^R e^{(a-s)t}dt$$

$$= \lim_{R\to\infty}\left(\frac{e^{(a-s)t}}{a-s}\right)\Bigg]_0^R$$

$$= \lim_{R\to\infty}\left(\underbrace{\frac{e^{(a-s)R}}{a-s}}_{\substack{\text{converges to } 0 \\ \text{if } -(a-s)>0 \\ s>a}} - \frac{e^0}{a-s}\right)$$

$$= -\frac{1}{a-s} = \frac{1}{s-a} \text{ if } s>a$$

So $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ if $s>a$

Note: $\mathcal{L}\{f(t)\pm g(t)\} = \mathcal{L}\{f(t)\}\pm\mathcal{L}\{g(t)\}$. We will see this via the next example.

Ex 4: Find the Laplace Transform of $f(t)=1+e^{-4t}$

$$\mathcal{L}\{1+e^{-4t}\} = \int_0^\infty (1+e^{-4t})e^{-st}dt = \int_0^\infty 1\cdot e^{-st}dt + \int_0^\infty e^{-4t}e^{-st}dt$$

Ex 4: Find the Laplace Transform of $f(t) = 1 + e^{-4t}$

$$\mathcal{L}\{1+e^{-4t}\} = \int_0^\infty (1+e^{-4t})e^{-st}\,dt = \int_0^\infty 1\cdot e^{-st}\,dt + \int_0^\infty e^{-4t}e^{-st}\,dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{e^{-4t}\}$$

$$= \frac{1}{s} + \frac{1}{s+4} \quad \text{if } s>0$$

Written HW: Let $f(t)$ and $g(t)$ be functions of t, and a be a constant. Proof the following

ⓐ $\mathcal{L}\{a f(t)\} = a\,\mathcal{L}\{f(t)\}$

ⓑ $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

ⓒ The Laplace Transform is linear.

Ex 5: Find the Laplace Transform of $f(t) = \sin(bt)$ for $t>0$.

$$\mathcal{L}\{\sin(bt)\} = \int_0^\infty \sin(bt)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R \sin(bt)e^{-st}\,dt$$

Let's look at the integral w/ limit for a sec.

$u = \sin(bt)$ $\quad dv = e^{-st}\,dt$

$du = \cos(bt)\cdot b\,dt$ $\quad v = \dfrac{e^{-st}}{-s}$

$$\int \sin(bt)e^{-st}\,dt = \frac{\sin(bt)e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\cos(bt)\cdot b\,dt$$

$$\int \sin(bt)e^{-st}\,dt = \frac{\sin(bt)e^{-st}}{-s} + \frac{b}{s}\int \cos(bt)e^{-st}\,dt$$

$u = \cos(bt)$ $\quad dv = e^{-st}\,dt$

$du = -\sin(bt)\cdot b\,dt$ $\quad v = \dfrac{e^{-st}}{-s}$

$$\int \sin(bt)e^{-st}\,dt = \frac{\sin(bt)e^{-st}}{-s} + \frac{b}{s}\left[\frac{\cos(bt)e^{-st}}{-s} - \int\left(\frac{e^{-st}}{-s}\right)(-\sin(bt)\cdot b)\,dt\right]$$

$$= \frac{\sin(bt)e^{-st}}{-s} + \frac{b}{s}\left[\frac{\cos(bt)e^{-st}}{-s} - \frac{b}{s}\int \sin(bt)e^{-st}\,dt\right]$$

$$= \frac{\sin(bt)e^{-st}}{-s} - \frac{b\cos(bt)e^{-st}}{s^2} - \frac{b^2}{s^2}\int \sin(bt)e^{-st}\,dt$$

$$= \frac{\sin(bt)e^{-st}}{-s} - \frac{b\cos(bt)e^{-st}}{s^2} - \frac{b^2}{s^2}\int \sin(bt)e^{-st}\,dt$$

Solve for the integral

$$\int \sin(bt)e^{-st}\,dt + \frac{b^2}{s^2}\int \sin(bt)e^{-st}\,dt = \frac{\sin(bt)e^{-st}}{-s} - \frac{b\cos(bt)e^{-st}}{s^2}$$

$$\left(1+\frac{b^2}{s^2}\right)\int \sin(bt)e^{-st}\,dt = \frac{(s)\sin(bt)e^{-st}}{-s^2} - \frac{b\cos(bt)e^{-st}}{s^2}$$

$$\frac{s^2+b^2}{s^2}\int \sin(bt)e^{-st}\,dt = -\frac{1}{s^2}\left((s)\sin(bt) - b\cos(bt)\right)e^{-st}$$

$$\int \sin(bt)e^{-st}\,dt = \frac{\cancel{s^2}}{s^2+b^2}\left(-\frac{1}{\cancel{s^2}}\right)\left((s)\sin(bt) - b\cos(bt)\right)e^{-st}$$

$$\int \sin(bt)e^{-st}\,dt = \frac{-e^{-st}}{s^2+b^2}\left((s)\sin(bt) - b\cos(bt)\right)$$

So back to the integral w/ bounds and the limit.

$$\lim_{R\to 0}\int_0^R \sin(bt)e^{-st}\,dt = \lim_{R\to 0}\left(\frac{-e^{-st}}{s^2+b^2}\left((s)\sin(bt) - b\cos(bt)\right)\right)\Bigg]_0^R$$

$$= \lim_{R\to 0}\left[\underbrace{\frac{e^{-sR}}{s^2+b^2}}_{\text{converges to 0 if } s>0}\left((s)\sin(Rb) - b\cos(bR)\right) - \frac{e^{-s\cdot 0}}{s^2+b^2}\left((s)\sin(0) - b\cos(0)\right)\right]$$

$$= \frac{-1}{s^2+b^2}(-b) = \frac{b}{s^2+b^2} \quad \text{if } s>0$$

So $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}$ if $s>0$

Similarly, we can find $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}$ if $s>0$.

Figure 1: Laplace Transform Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n	$\dfrac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right) \quad c > 0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$