Ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad \underline{\underline{C}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\underset{(2\times3)\ (2\times2)}{\underline{\underline{A}}\,\underline{\underline{C}}} = \times$ NOT Defined

$\times$

$\checkmark$

$\underset{(2\times2)(2\times3)}{\underline{\underline{C}}\,\underline{\underline{A}}} = \begin{bmatrix} & & \\ & & \end{bmatrix} (2\times3)$ this is defined

Poll: $\underline{\underline{A}} = \overset{(3\times3)}{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}$ $\quad \underline{\underline{B}} = \overset{(2\times3)}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}$

Which of the following matrix mult. is valid?

$\times$ (a) $\underline{\underline{A}}\,\underline{\underline{B}}$ $(3\times3)(2\times3)$ $\qquad$ (c) neither

$\checkmark$

(b) $\underline{\underline{B}}\,\underline{\underline{A}}$ $(2\times3)(3\times3)$ $\qquad$ (d) both $\times$

III. Determinant:

If $\underline{\underline{A}}$ is a 2×2 matrix then its determinant

$$\det(\underline{\underline{A}}) = |\underline{\underline{A}}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex: $\det\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1\cdot3 - 2\cdot4 = 3-8 = -5$

For determinant of bigger square matrices, we use Expansion by Minors

If $\underline{\underline{A}}$ is an $n\times n$ matrix

$(n-1)\times(n-1)$ matrix obtained by

If $\underline{A}$ is an $n \times n$ matrix

Let $\underline{A}_{ij}$ be the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column

$$\det(\underline{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\underline{A}_{ij})$$

Ex: $\underline{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix}$ let $i=1$

$$\det(\underline{A}) = (-1)^{1+1} a_{11} \det(\underline{A}_{11}) + (-1)^{1+2} a_{12} \det(\underline{A}_{12}) + (-1)^{1+3} a_{13} \det(\underline{A}_{13})$$

$$= (-1)^2 (3) \det\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^3 (1) \det\begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} + (-1)^4 (-2) \det\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= 3(2\cdot 5 - 1\cdot 3) + (-1)(4\cdot 5 - 1(-2)) + (-2)(4\cdot 3 - 2(-2))$$

$$\boxed{\det(\underline{A}) = -33}$$

IV, Matrix-Valued Functions:

Def: A matrix-valued function is a matrix in which each element is a function of t

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ 3t^2 \end{bmatrix}$ $\underline{A}(t) = \begin{bmatrix} \sin(t) & 7 \\ 0 & 8t^2+1 \end{bmatrix}$

matrix fcn $\underline{A}(t)$ is continuous

Def: We say that a matrix fcn $\underline{A}(t)$ is <u>continuous</u> at a point t if each of its elements is continuous

Def: The <u>derivative</u> of a matrix fcn is defined by elementwise differentiation

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ $\quad \underline{x}'(t) = \frac{d\underline{x}}{dt} = \begin{bmatrix} 1 \\ 2t \\ -e^{-t} \end{bmatrix}$

sum and product rules

$$\frac{d}{dt}\left(\underline{\underline{A}}(t) + \underline{\underline{B}}(t)\right) = \underline{\underline{A}}' + \underline{\underline{B}}'$$

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \underline{\underline{A}}\,\underline{\underline{B}}' + \underline{\underline{A}}'\underline{\underline{B}}$$ HW problem to show this

$$\frac{d}{dt}\left(c\,\underline{\underline{A}}(t)\right) = c\,\underline{\underline{A}}'$$ c is a scalar const (doesn't depend on t)

V First Order Linear Systems:

system of ODEs $\longrightarrow$ matrix form

$$\frac{d\underline{x}}{dt} = \underbrace{\underline{\underline{P}}(t)}_{\text{coefficient matrix}}\,\underline{x} + \underline{f}(t)$$

$(n\times 1)$ $\quad (n\times n)(n\times 1)$ $\quad (n\times 1)$

Ex: 1st order system

$$x_1' = 4x_1 - 3x_2$$
$$x_2' = 6x_1 - 7x_2$$

matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Verify $x(t) = \begin{bmatrix} 3e^{2t} \\ \ldots e^{2t} \end{bmatrix}$ is a soln

Verify $\underline{x}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ is a soln

$$\underline{x}' \overset{?}{=} \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \underline{x}$$

VI. Associated Homogeneous Eqn

when $\underline{f}(t) = 0$ the eqn is homogeneous

$$(*) \qquad \underline{x}' = \underline{\underline{P}}(t)\,\underline{x}$$

If $\underline{\underline{P}}(t)$ is an $n \times n$ matrix, then we expect $(*)$ to have n linearly independent solutions

$$\underline{x}_1(t),\ \underline{x}_2(t),\ \dots,\ x_n(t)$$

Since the system is linear, the Principle of Superposition applies

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \dots + c_n \underline{x}_n(t)$$

where $c_1, c_2, \dots, c_n$ are scalar constants

then $\underline{x}(t)$ also solves $(*)$

Ex: $\underline{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \underline{x}$

has solutions $\underline{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ $\quad \underline{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

Then the general solution is

$$\underline{x}(t) = c_1\, \underline{x}(t) + c_2\, \underline{x}_2(t)$$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ \dots \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ \dots \end{bmatrix}$$

$\underline{x}(t) = c_1 \ldots$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

VII Linear Independence: Wronskian

If $\underline{x}_1, \ldots, \underline{x}_n$ are solutions to ODE $(*)$

the Wronskian is

$$W(t) = \det \begin{bmatrix} | & | & & | \\ \underline{x}_1(t) & \underline{x}_2(t) & \cdots & x_n(t) \\ | & | & & | \end{bmatrix} \quad (n \times n)$$

If $W(t) \neq 0$ then are linearly independent

Ex:

$$W(t) = \det \begin{bmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{bmatrix} = (3e^{2t})(3e^{-5t}) - (2e^{2t})(e^{-5t})$$

$$= 9e^{-3t} - 2e^{-3t}$$

$$= 7e^{-3t} \neq 0$$

so $\underline{x}_1$ and $\underline{x}_2$ are linearly independent

☆ Summary:

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
- use expansion by minors for determinants of $n \times n$ matrices
- ... functions — take derivatives elementwise

- matrix-valued functions — take derivatives elementwise

- Homogeneous Linear Systems

$$\underline{x}' = \underline{\underline{P}}(t)\underline{x} \qquad \text{where } \underline{\underline{P}}(t) \text{ is } n \times n$$

have n linearly independent solutions

$$\underline{x}_1(t), \underline{x}_2(t), \ldots, \underline{x}_n(t)$$

use the Principle of superposition to get the general solution:

$$\underline{x}(t) = c_1 \underline{x}_1(t) + \ldots + c_n \underline{x}_n(t)$$

- Use the Wronskian to determine linear independence.