Recall from last class the Laplace Transform for $t>0$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

And we have the following table

Figure 1: Laplace Transform Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	t^p $(p>-1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$ $c>0$
16.	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Good news Laplace Transforms also work for piecewise continuous functions!!!

Ex 6: Find the Laplace Transform of

$$f(t) = \begin{cases} -1, & 0<t<2 \\ 1 & t>2 \end{cases}$$

So
$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}dt = \int_0^2 -1e^{-st}dt + \int_2^\infty 1\cdot e^{-st}dt$$
$$= \int_0^2 -e^{-st}dt + \lim_{R\to\infty}\int_2^R e^{-st}dt$$

$$= \int_0^2 -e^{-st}dt + \lim_{R\to\infty}\int_2^R e^{-st}dt$$

$$= \frac{+e^{-st}}{+s}\Big]_0^2 + \lim_{R\to\infty}\left(\frac{e^{-st}}{-s}\right)\Big]_2^R$$

$$= \frac{e^{-s(2)}}{s} - \frac{e^{-s(0)}}{2} + \lim_{R\to\infty}\left(\underbrace{\frac{e^{-sR}}{-s}}_{\text{converges to 0 if } s>0} - \frac{e^{-2s}}{-s}\right)$$

$$= \frac{e^{-2s}}{s} - \frac{1}{2} + \frac{e^{-2s}}{s}$$

$$= \frac{2e^{-2s}}{s} - \frac{1}{2}$$

Inverse Laplace Transforms

If $F(s) = \mathcal{L}\{f(t)\}$ then we call $f(t)$ the Inverse Laplace transform of $f(s)$.

i.e. If $F(s) = \mathcal{L}\{f(t)\}$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

Ex 7: We previously found that

ⓐ $\mathcal{L}\{3\} = \frac{3}{s}$

ⓑ $\mathcal{L}\{e^{4t}\} = \frac{1}{s+4}$

ⓒ $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$

So what are the inverse Laplace Transforms?

ⓐ $\mathcal{L}^{-1}\{\frac{3}{s}\} = 3$

ⓑ $\mathcal{L}^{-1}\{\frac{1}{s+4}\} = e^{4t}$

ⓒ $\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} = \sin(t)$

Ex 8: Find the inverse Laplace Transform of $F(s) = \frac{2}{s} + \frac{3}{s-5}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{3}{s-5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

$$= 2\cdot 1 + 3\cdot e^{5t}$$

$$= 2 + 3e^{5t}$$

Ex 9: Find the inverse Laplace Transform of $F(s) = \frac{1}{s^2+6s+13}$

Note: $s^2+6s+13$ is irreducible so let's rewrite it to look like

Note: $s^2+6s+13$ is irreducible so let's rewrite it to look like
$$(s-a)^2+b^2$$

So let's complete the square.
$$s^2+6s+9+13-9=(s+3)^2+4=(s-(-3))^2+2^2$$

So $\dfrac{1}{s^2+6s+13}=\dfrac{1}{(s-(-3))^2+2^2}$ looks like $\dfrac{b}{(s-a)^2+b^2}$

All we are missing is a 2 in the numerator.
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+13}\right\}=\mathcal{L}^{-1}\left\{\frac{2}{2}\cdot\frac{1}{(s-(-3))^2+2^2}\right\}$$
by the table
$$=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-(-3))^2+2^2}\right\}=\frac{1}{2}e^{-3t}\sin(2t)$$

The method of partial fractions is often helpful in finding Laplace inverse transforms.

Ex 10: Find the inverse Laplace Transform of $F(s)=\dfrac{4s-11}{s^2-7s+10}$

Note the denominator is factorable.
$$s^2-7s+10=(s-2)(s-5)$$

So we can use partial fraction decomposition.
$$\frac{4s-11}{s^2-7s+10}=\frac{A}{s-2}+\frac{B}{s-5}=\frac{A(s-5)+B(s-2)}{(s-2)(s-5)}$$
$$4s-11=As-5A+Bs-2B$$
$$4s-11=(A+B)s-(5A+2B)$$
$$\Rightarrow\begin{cases}A+B=4 & ①\\ 5A+2B=11 & ②\end{cases}$$

Multiply ① by 2 and perform ①-② | Plug $A=1$ into ①
$A+B=4$

Multiply ① by 2 and perform ①-②

$$2A+\cancel{2B}=8$$
$$-(5A+\cancel{2B}=11)$$
$$-3A=-3$$
$$A=1$$

Plug $A=1$ into ①

$$A+B=4$$
$$1+B=4$$
$$B=3$$

So $\dfrac{4s-11}{s^2-7s+10}=\dfrac{1}{s-2}+\dfrac{3}{s-5}$

So $\mathcal{L}^{-1}\left\{\dfrac{4s-11}{s^2-7s+10}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\}+3\mathcal{L}^{-1}\left\{\dfrac{1}{s-5}\right\}$

$$=e^{2t}+3e^{5t}$$