Recall from Last class we defined the Laplace Transform of a function $f(t)$ as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-st} f(t)\,dt$$

and the inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Also we have a table of common Laplace Transforms.

Now we are going to use all of that to solve IVP (e.g. mass spring problem)

Consider the IVP:

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = x_0 \\ x'(0) = x_0' \end{cases}$$

First start by taking the Laplace transform of the ODE

$$\mathcal{L}\{ax'' + bx' + cx\} = \mathcal{L}\{f(t)\}$$

By linearity,

$$a\underbrace{\mathcal{L}\{x''\}} + b\underbrace{\mathcal{L}\{x'\}}_{\text{But what are these?}} + c\underbrace{\mathcal{L}\{x\}}_{\text{by definition } = X(x)} = \mathcal{L}\{f(t)\}$$

Let's figure out $\mathcal{L}\{x'\}$

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st} x'(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-st} x'(t)\,dt$$

$u = e^{-st}$ $\qquad dv = x'(t)\,dt$

$du = -se^{-st}dt$ $\qquad v = x(t)$

$u = e$
$du = -se^{-st}dt \qquad v = x(t)$

$$= \lim_{R\to\infty}\left(x(t)e^{-st}\Big]_0^R - \int_0^R x(t)(-s)e^{-st}dt \right)$$

$$= \lim_{R\to\infty}\left(\underbrace{x(R)e^{-sR}}_{\text{converges to 0 when } s>0} - x(0)e^{-0} + s\int_0^R x(t)e^{-st}dt \right)$$

$$= -x(0) + s\int_0^\infty x(t)e^{-st}dt$$

$$= -x(0) + s\,\mathcal{L}\{x(t)\}$$

$$\mathcal{L}\{x'\} = -x(0) + s\,X(s) \quad \text{[by def of Laplace Transform]}$$

We can repeat this procedure for higher order derivatives

$$\mathcal{L}\{x''\} = \mathcal{L}\{(x')'\} = -x'(0) + s\,\mathcal{L}\{x'\}$$
$$= -x'(0) + s\left[-x(0) + sX(s)\right]$$
$$= -x'(0) - s\cdot x(0) + s^2 X(s)$$

Note you can repeat this process for $\mathcal{L}\{x'''\}$, $\mathcal{L}\{x^{(4)}\}$, and so on.

So in general to solve linear ODEs using Laplace Transforms

① Take the Laplace transform of both sides of the ODE, and separate the terms using the property of linearity.

② Replace the terms, where applicable, with the formulas we found.

$$\mathcal{L}\{x'\} = -x(0) + sX(s) \qquad \text{and } \mathcal{L}\{x\} = X(s)$$
$$\mathcal{L}\{x''\} = -x'(0) - s\cdot x(0) + s^2X(s)$$

and plug in the initial conditions

③ Solve the eqn for $X(s)$

and plug in the initial conditions

③ Solve the eqn for $\mathcal{X}(s)$.

④ Using our table of Laplace transform, determine the solution

$$x = \mathcal{L}^{-1}\{\mathcal{X}(s)\}$$

Ex 1: Solve the IVP

$$\begin{cases} x'' - x' - 6x = 0 \\ x(0) = 2 \\ x'(0) = -1 \end{cases}$$

Step 1: $\mathcal{L}\{x'' - x' - 6x\} = \mathcal{L}\{0\}$

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = 0$$

Step 2: $(s^2\mathcal{X}(s) - s\cdot x(0) - x'(0)) - (s\cdot\mathcal{X}(s) - x(0)) - 6\mathcal{X}(s) = 0$

$$s^2\mathcal{X}(s) - s(2) - (-1) - (s\cdot\mathcal{X}(s) - 2) - 6\mathcal{X}(s) = 0$$

$$s^2\mathcal{X}(s) - 2s + 1 - s\mathcal{X}(s) + 2 - 6\mathcal{X}(s) = 0$$

Step 3: $s^2\mathcal{X}(s) - s\mathcal{X}(s) - 6\mathcal{X}(s) = 2s - 3$

$$\mathcal{X}(s)[s^2 - s - 6] = 2s - 3$$

$$\mathcal{X}(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)}$$

Step 4: $x(t) = \mathcal{L}^{-1}\{\mathcal{X}(s)\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-3)(s+2)}\right\}$

Note to find the inverse Laplace transform we need to use partial fraction decomposition.

$$\frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{A(s+2)+B(s-3)}{(s-3)(s+2)}$$

$$2s - 3 = (A+B)s + (2A - 3B)$$

$$\Rightarrow \begin{cases} 2 = A + B & ① \\ -3 = 2A - 3B & ② \end{cases}$$

Multiply ① by 3 and then ①+② | Plug $A = 3/5$ into ①

$-3 = 2A - 3B$ ②

Multiply ① by 3 and then ①+②

$$\begin{array}{r} 6 = 3A + \cancel{3B} \\ +(-3 = 2A - \cancel{3B}\;) \\ \hline 3 = 5A \end{array}$$

$$A = \frac{3}{5}$$

Plug $A = 3/5$ into ①

$$2 = \frac{3}{5} + B$$

$$10 = 3 + 5B$$

$$7 = 5B$$

$$B = \frac{7}{5}$$

So $\dfrac{2s-3}{(s-3)(s+2)} = \dfrac{3}{5} \cdot \dfrac{1}{s-3} + \dfrac{7}{5} \cdot \dfrac{1}{s+2}$

Hence $x(t) = \dfrac{3}{5} \mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} + \dfrac{7}{5} \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$

$$= \frac{3}{5} e^{3t} + \frac{7}{5} e^{-2t}$$

<u>Ex 2</u>. Solve the IVP

$$\begin{cases} x'' + 4x = \sin(3t) \\ x(0) + x'(0) = 0 \end{cases}$$

<u>Step 1</u>: $\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin(3t)\}$

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\sin(3t)\}$$

<u>Step 2</u>: $\left(s^2 X(s) - s\,x(0) - x'(0)\right) + 4X(s) = \dfrac{3}{s^2+9}$

$$\left(s^2 X(s) - s \cdot 0 - 0\right) + 4X(s) = \frac{3}{s^2+9}$$

<u>Step 3</u>: $s^2 X(s) + 4X(s) = \dfrac{3}{s^2+9}$

$$X(s)\left[s^2+4\right] = \frac{3}{s^2+9}$$

$$X(s) = \frac{3}{(s^2+4)(s^2+9)}$$

<u>Step 4</u>: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3}{(s^2+4)(s^2+9)}\right\}$

Note to find the inverse Laplace transform of this we need to use partial fraction decomposition.

Step 4: $x(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+4)(s^2+9)}\right\}$ partial fraction decomposition.

$$\frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9} = \frac{(As+B)(s^2+9)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+9)}$$

	As	B
s^2	As^3	Bs^2
9	$9As$	$9B$

	Cs	D
s^2	Cs^3	Ds^2
4	$4Cs$	$4D$

$$3 = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D)$$

$$\Rightarrow \begin{cases} A+C=0 & (1) \\ B+D=0 & (2) \\ 9A+4C=0 & (3) \\ 9B+4D=3 & (4) \end{cases}$$

First look at eqns (1) and (3).

$$A+C=0 \quad (1)$$
$$9A+4C=0 \quad (3)$$

If we solve (1) for A,

$$A=-C$$

Plug that into (3)

$$9(-C)+4C=0$$
$$-5C=0$$
$$C=0$$

which implies $A=0$

Note eqns (2) and (4)

$$B+D=0 \quad (2)$$
$$9B+4D=3 \quad (4)$$

Solve (2) for B,

$$B=-D$$

Plug that into (4).

$$9(-D)+4D=3$$
$$-9D+4D=3$$
$$-5D=3$$
$$D=-3/5$$

Plug $D=-3/5$ into $B=-D$

$$B=3/5$$

So $x(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+4)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{5}\cdot\frac{1}{s^2+4} - \frac{3}{5}\cdot\frac{1}{s^2+9}\right\}$

$$= \frac{3}{5}\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+4}}\right\} - \frac{3}{5}\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+9}}\right\}$$

Note $a=2$ and we need 2 in the numerator

based on the table

Note $a=3$ and we need 3 in the numerator

$$= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{2}{2}\cdot\frac{1}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{2}{2}\cdot\frac{1}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$

Recap: Laplace Transform Procedure

variable t — variable s

ODE for $x(t)$ $\rightarrow$ $\mathcal{L}$ $\rightarrow$ algebraic eqn for $X(s)$

$\downarrow$

Solve using algebra (like partial fractions)

$\downarrow$

$X(s)$ $\rightarrow$ $\mathcal{L}^{-1}$ $\rightarrow$ Solution $x(t)$

Transform Theorems:

We have showed that

Thm 1: (Transform of Derivatives)

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

Background work

t	s
derivative in t	multiplication by s

$$\frac{d}{dt}(e^{st}) = s(e^{st})$$

Similary

Thm 2: (Transforms of Integrals)

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

t	s
integral in t	division by s

$$\int e^{st}dt = \frac{e^{st}}{s}$$

By Thm 2, we can find the inverse Laplace Transforms

Ex 3: Find the inverse Laplace Transform of $G(s) = \frac{1}{s(s-3)}$

Ex 3: Find the inverse Laplace Transform of $G(s) = \dfrac{1}{s(s-3)}$

First let's do some rewriting so we get $G(s) = \dfrac{F(s)}{s}$

$$G(s) = \frac{1}{s} \cdot \frac{1}{s-3} = \frac{1}{s} \cdot F(s) = \frac{F(s)}{s} \quad \text{if} \quad F(s) = \frac{1}{s-3}$$

By the Table,

$$F(s) = \frac{1}{s-3} \Rightarrow f(t) = e^{3t}$$

By theorem 2

$$\text{So } g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t e^{3t}\,dt$$

$$= \frac{e^{3t}}{3}\Big]_0^t = \frac{e^{3t}}{3} - \frac{e^0}{3} = \frac{1}{3}(e^t - 1)$$