Section 7.3: Translation and Partial Fractions

Saturday, November 8, 2025 11:00 AM

Recap: Laplace Transform Procedure

variable t — variable s

ODE for $x(t)$ $\rightarrow$ $\mathcal{L}$ $\rightarrow$ algebraic eqn for $X(s)$

$\downarrow$

Solve using algebra (like partial fractions)

$\downarrow$

Solution $x(t)$ $\leftarrow$ $\mathcal{L}^{-1}$ $\leftarrow$ $X(s)$

KEY STEP: Arrange $X(s)$ so that we can take $\mathcal{L}^{-1}\{X(s)\} = x(t)$

Theorem 1: (Translation on the s-axis)

If $F(s) = \mathcal{L}\{f(t)\}$ exists, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Equivalently $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$

Background Idea

t	s
Multiplying by e^{at}	then $s \mapsto s-a$

$$e^{at}e^{-st} = e^{-(s-a)t}$$

Ex 1: Find the Laplace Transform of $g(t) = e^{3t}\sin(\pi t)$

$$\mathcal{L}\{\underbrace{e^{3t}\sin(\pi t)}\} = F(s-3) \quad \text{with } a = 3$$

This is in form we want for Thm 1

Hence $f(t) = \sin(\pi t)$. So $F(s) = \mathcal{L}\{\sin(\pi t)\} = \dfrac{\pi}{s^2+\pi^2}$ (by the Table)

$$F(s-3) = \frac{\pi}{(s-3)^2+\pi^2}$$

Hence $\mathcal{L}\{e^{3t}\sin(\pi t)\} = \dfrac{\pi}{(s-3)^2+\pi^2}$

Ex 2: Find the inverse Laplace Transform of $G(s) = \dfrac{2s+3}{\ldots}$

Ex 2: Find the inverse Laplace Transform of $G(s) = \dfrac{2s+3}{s^2+2s+5}$

Note s^2+2s+5 can't be factored. BUT to use our table we still need to do some rewriting. Let's complete the square

$$s^2+2s+1+5-1 = (s+1)^2+4 = (s+1)^2+2^2$$

So $$\frac{2s+3}{s^2+2s+5} = \frac{2s+3}{(s+1)^2+2^2}$$

Note we want to split the fraction to have either (and/or both)

$$\frac{(s+1)}{(s+1)^2+2^2} \quad \text{or} \quad \frac{2}{(s+1)^2+2^2}$$

So $2s+3 = 2s+2+1 = 2(s+1)+1$

So $$\frac{2s+3}{(s+1)^2+2^2} = \frac{2(s+1)}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}$$

Almost there we need a 2 instead of 1. So, multiply by $\frac{2}{2}$ this fraction.

$$= 2\cdot\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

So $$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+2s+5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

$$= 2\cos(2t) + \frac{1}{2}\sin(2t)$$

Ex 3: Solve the IVP

$$\begin{cases} y''-4y'+3y = t^2 \\ y(0)=y'(0)=0 \end{cases}$$

Recall from Last class

$$\mathcal{L}\{y''\} = -y'(0) - s\cdot y(0) + s^2Y(s) \qquad \mathcal{L}\{y\} = Y(s)$$

$\mathcal{L}\{y''\} = -y'(0) - s \cdot y(0) + s^2 Y(s)$ $\quad$ $\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = -y(0) - s\, Y(s)$

So with our initial conditions: $\mathcal{L}\{y''\} = s^2 Y(s)$ and $\mathcal{L}\{y'\} = s\,Y(s)$

Take the Laplace of both sides of our ODE.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{t^2\}$$

So $s^2 Y(s) - 4s\,Y(s) + 3\,Y(s) = \dfrac{2!}{s^{2+1}}$ From the table.

$$Y(s)\left[s^2 - 4s + 3\right] = \frac{2}{s^3}$$

$$Y(s) = \frac{2}{s^3(s^2 - 4s + 3)} = \frac{2}{s^3(s-1)(s-2)}$$

Now let's expand $Y(s)$, using partial fraction decomposition.

$$\frac{2}{s^3(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1} + \frac{E}{s-3}$$

$$= \frac{As^2(s-1)(s-3) + Bs(s-1)(s-3) + C(s-1)(s-3) + Ds^3(s-3) + Es^3(s-1)}{s^3(s-1)(s-3)}$$

$$2 = As^2(s^2 - 4s + 3) + Bs(s^2 - 4s + 3) + C(s^2 - 4s + 3) + D s^3(s-2) + E s^3(s-1)$$

s^4 terms $\quad 0 = A + D \quad$ ①

s^3 terms $\quad 0 = -4A + B - 2D - E \quad$ ②

s^2 terms $\quad 0 = 3A - 4B + C \quad$ ③

s terms $\quad 0 = 3B - 4C \quad$ ④

constant terms $\quad 2 = 3C \quad$ ⑤

Solve ⑤ for C.

$C = \dfrac{2}{3}$

Plug $C = 3/2$ in ④

$0 = 3B - 4\left(\frac{2}{3}\right)$

$0 = 3B - \frac{8}{3}$

$B = \dfrac{8}{9}$

Plug $C = 2/3$ and $B = 8/9$ in ③

$0 = 3A - 4(8/9) + 2/3$

$0 = 3A - 32/9 + 6/9$

$0 = 3A - 26/9$

$\frac{26}{9} = 3A$

$B = \frac{8}{9}$

$\frac{26}{9} = 3A$

$A = {}^{26}/_{27}$

Plug $A = {}^{26}/_{27}$ into ①.

$${}^{26}/_{27} + D = 0$$

$$D = -{}^{26}/_{27}$$

Plug $A = {}^{26}/_{27}$, $B = \frac{8}{9}$, $D = -{}^{26}/_{27}$ into ②.

$$-4\left(\frac{26}{27}\right) + \frac{8}{9} - 2\left(-\frac{26}{27}\right) - E = 0$$

$$-4\left(\frac{26}{27}\right) + 2\left(\frac{26}{27}\right) + \frac{8}{9} = E$$

$$-2\left(\frac{26}{27}\right) + \frac{24}{27} = E$$

$$-\frac{28}{27} = E$$

So $$\frac{2}{s^3(s-1)(s-3)} = \frac{26}{27} \cdot \frac{1}{s} + \frac{8}{9} \cdot \frac{1}{s^2} + \frac{2}{3} \cdot \frac{1}{s^3} - \frac{26}{27} \cdot \frac{1}{s-1} - \frac{28}{27} \cdot \frac{1}{s-3}$$

$$y(s) = \mathcal{L}^{-1}\left\{\frac{2}{s^3(s-1)(s-3)}\right\} = \frac{26}{27}\mathcal{L}^{-1}\left\{{}^{1}/_{s}\right\} + \frac{8}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$- \frac{26}{27}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{28}{27} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

By the table

$$= \frac{26}{27} \cdot 1 + \frac{8}{9} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} - \frac{26}{27}e^{t} - \frac{28}{27}e^{3t}$$

All we are missing is

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

Remember $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

So $\frac{1}{s^2} = \frac{1}{s^{1+1}} = \frac{1!}{s^{1+1}}$ so $n=1$ $\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\frac{1}{s^3} = \frac{1}{s^{2+1}} = \frac{2}{2} \cdot \frac{1}{s^{2+1}} = \frac{1}{2} \cdot \frac{2!}{s^{2+1}}$ so $n=2$ $\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = \frac{1}{2}t^2$

$$\frac{1}{s^3} = \frac{1}{s^{2+1}} = \frac{2}{2} \cdot \frac{1}{s^{2+1}} = \frac{1}{2} \cdot \frac{2!}{s^{2+1}} \quad \text{so } n=2 \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = \frac{1}{2} t$$

Now,

$$y(s) = \frac{26}{27} + \frac{8}{9} \cdot t + \frac{2}{3} \cdot \frac{1}{2} t^2 - \frac{26}{27} e^{t} - \frac{28}{27} e^{3t}$$