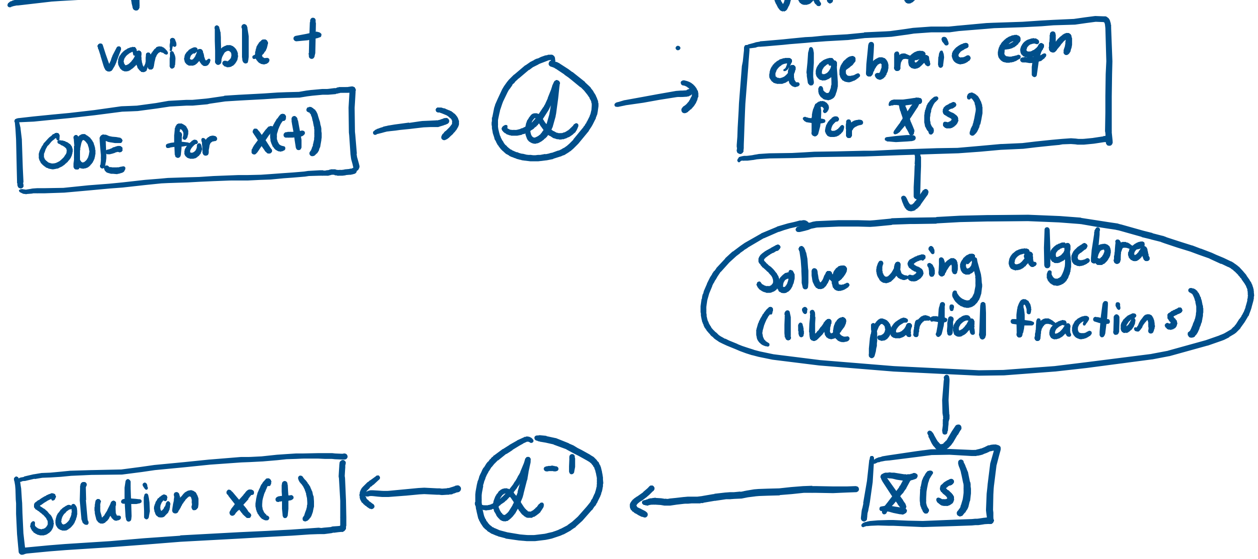
Recap: Laplace Transform Procedure



Our solution $X(s)$ often has the form
$$X(s) = F(s)\cdot G(s)$$
Take the Laplace Transform of that and we get
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\}$$
So can $\mathcal{L}^{-1}\{F(s)\}\cdot\mathcal{L}\{G(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\}$?

Suppose we have the IVP $\begin{cases} x'' + x = \cos(t) \\ x(0) = x'(0) = 0 \end{cases}$

We get $X(s) = \dfrac{s}{(s^2+1)^2} = \dfrac{s}{s^2+1}\cdot\dfrac{1}{s^2+1} = \mathcal{L}\{\cos(t)\}\,\mathcal{L}\{\sin(t)\}$

Let's see if $\dfrac{s}{(s^2+1)^2} = \mathcal{L}\{\cos(t)\sin(t)\}$.

Well $\mathcal{L}\{\cos(t)\sin(t)\} = \mathcal{L}\{\tfrac{1}{2}\sin(2t)\}$ (double angle formula)
$$= \tfrac{1}{2}\mathcal{L}\{\sin(2t)\}$$

$$= \frac{1}{2}\mathcal{L}\{\sin(2t)\}$$

$$= \frac{1}{2}\cdot\frac{2}{s^2+2^2} \qquad \text{(by the table)}$$

$$= \frac{1}{s^2+4}$$

Hence does $\frac{1}{s^2+4} \overset{?}{=} \frac{s}{(s^2+1)^2}$ <u>NO</u>!!!

Hence $\mathcal{L}\{F(s)\cdot G(s)\} \neq \mathcal{L}\{F(s)\}\cdot\mathcal{L}\{G(s)\}$

So do there exist a formula for the product of Laplace Transforms?

<u>YES!</u>

Before we state the formula we need one more thing

<u>Def:</u> The <u>convolution</u> of two functions $f(t)$ and $g(t)$ is defined by

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du$$

<u>Written HW:</u> Prove that $(f*g) = (g*f)$

i.e. order of f and g doesn't matter.

<u>Thm 1:</u> (Convolution Property)

For two functions $f(t)$ and $g(t)$

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

and equivalently

$$(f*g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$$

<u>Ex 1:</u> Find the inverse Laplace Transform of $H(s) = \frac{2}{(s+1)(s-3)}$

(... for this problem)

Ex 1. Find $\qquad (s+1)(s-3)$

(Note we can use partial fractions for this problem)

Using the convolution property,

$$H(s) = \underbrace{\left(\frac{2}{s+1}\right)}_{F(s)}\underbrace{\left(\frac{1}{s-3}\right)}_{G(s)}$$

So $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 2e^{-t}$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

So $\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} \overset{\text{By Thm}}{=} (f*g)(t)$

$$= \int_0^t f(u)g(t-u)\,du$$

$$= \int_0^t 2e^{-u}e^{3(t-u)}\,du$$

$$= \int_0^t 2e^{-u}e^{3t}e^{-3u}\,du$$

$$= 2e^{3t}\int_0^t e^{-4u}\,du$$

$$= 2e^{3t}\left(\frac{e^{-4u}}{-4}\right)\Bigg]_0^t$$

$$= \frac{2e^{3t}}{-4}\left(e^{-4t}-1\right)$$

$$= -\frac{1}{2}\left(e^{-t}-e^{3t}\right)$$

Note we can also the convolution property to write down solutions to ODE.

Ex 2: Solve the IVP:

Ex 2: Solve the IVP:

$$\begin{cases} x'' + x = \cos(t) \\ x(0) = x'(0) = 0 \end{cases}$$

In the beginning of class we said

$$X(s) = \frac{s}{(s^2+1)^2} = \underbrace{\frac{s}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{G(s)}$$

So $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$

So $\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$ (by Thm)

$$= \int_0^t f(w)g(t-w)\,dw$$

$$= \int_0^t \cos(w)\sin(t-w)\,dw$$

$$= \int_0^t \frac{1}{2}\left[\sin(t) + \sin(t-2w)\right]dw$$

$$= \frac{1}{2}\sin(t)\int_0^t dw + \frac{1}{2}\int_0^t \sin(t-2w)\,dw$$

$$= \frac{1}{2}\sin(t)\,(w)\Big]_0^t + \frac{1}{2}\cdot\frac{-\cos(t-2w)}{-2}\Big]_0^t$$

$$= \frac{1}{2}\sin(t)\cdot t + \frac{1}{4}\left(\cos(t-2t) - \cos(t)\right)$$

$$= \frac{t}{2}\sin(t) + \frac{1}{4}\left(\underbrace{\cos(-t) - \cos(t)}_{\to 0}\right)$$

Remember $\cos(-t) = \cos(t)$

$$= \frac{t}{2}\sin(t)$$

Now Differentiation of Transforms

Thm 2: (Differentiation of Transforms) Let $f(t)$ be a function, then

$$\mathcal{L}\{-t f(t)\} = F'(s)$$

which equivalently

$$-t f(t) = \mathcal{L}^{-1}\{F'(s)\} \iff f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

Note: If we repeatedly use this formula we get

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Ex 3: Find $\mathcal{L}\{t^2 \sin(\pi t)\}$

Note $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

In our case $n=2$ and $f(t) = \sin(\pi t)$. So

$$\mathcal{L}\{t^2 \sin(\pi t)\} = (-1)^2 F^{(2)}(s)$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}^{-1}\{\sin(\pi t)\}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{\pi}{s^2+\pi^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-\pi}{(s^2+\pi^2)^2}\,2s\right)$$

$$= \frac{d}{ds}\left(\frac{-2\pi s}{(s^2+\pi^2)^2}\right)$$

$$= \frac{-2\pi(s^2+\pi^2)^{\cancel{2}} + (+2\pi s)\cdot 2\cancel{(s^2+\pi^2)}\cdot 2s}{(s^2+\pi^2)^{\cancel{4}\,3}}$$

$$(s^2+\pi^2)^{\cancel{4}\,3}$$

$$= \frac{2\pi[-s^2-\pi^2+2s^2]}{(s^2+\pi^2)^3} = \frac{2\pi[s^2-\pi^2]}{(s^2+\pi^2)^3}$$

Lastly integration with transforms

Thm 3: (Integration with transforms) Let $f(t)$ be a function

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

and equivalently

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\int_s^\infty F(u)\,du\right\}$$

$$f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(u)\,du\right\}$$

Ex 4: Find $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\}$

Note that $f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_0^\infty F(u)\,du\right\}$

In our case $F(s) = \frac{2s}{(s^2-1)^2}$

So $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2u}{(u^2-1)^2}\,du\right\}$

$$= t\,\mathcal{L}^{-1}\left\{\lim_{R\to\infty}\int_s^R \frac{2u}{(u^2-1)^2}\,du\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \int_s \frac{u}{(u^2-1)^2}\,du \right\}$$

$a = u^2-1$

$da = 2u\,du$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \int \frac{da}{a^2} \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \int a^{-2}\,da \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \frac{a^{-1}}{-1} \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \left(\frac{-1}{u^2-1} \right) \Big]_s^R \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \lim_{R\to\infty} \underbrace{\frac{-1}{R^2-1}}_{=0} - \frac{-1}{s^2-1} \right\}$$

$$= t\,\mathcal{L}^{-1}\left\{ \frac{1}{s^2-1} \right\}$$

by the table

$$= t\,\sinh(t)$$