# Section 7.5: Periodic and Piecewise Continuous functions

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We are solving ODEs of the form

$$ax'' + bx' + cx = f(t)$$

using Laplace Transforms. One advantage of Laplace Transforms is we can solve these ODEs where $f(t)$ is piecewise continuous.

Why is this important?

There are alot of mathematical models of mechanical or electrical Systems often involve functions with discontinuities corresponding to external forces that turn on and off by external forces.

A simple on-off function is the unit step function, which is

$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$


So what is $\mathcal{L}\{u(t-a)\}$?

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt$$

$$= \int_0^\infty e^{-st}\,dt \quad \text{if } t \geq a$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \lim_{R\to\infty} \int_a^R e^{-st}\,dt$$

$$= \lim_{R\to\infty} \left.\frac{e^{-st}}{-s}\right]_a^R$$

$$= \lim_{R\to\infty} \left( \underbrace{\frac{e^{-sR}}{-s}}_{\to 0} - \left(\frac{e^{-sa}}{-s}\right)\right) = \frac{e^{-sa}}{s} = e^{-sa}\left(\frac{1}{s}\right) = e^{-sa}\underbrace{\mathcal{L}\{1\}}$$

Thm: (Translation on the $t$-axis)

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-sa}F(s)$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e$$

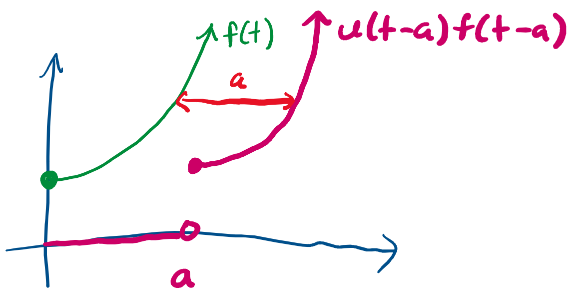
and equivalently

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}F(s)\}$$

for $s > c+a$.

<u>Note:</u> $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$ So $u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a \end{cases}$

<u>Why?</u>




$u(t-a)f(t-a)$ denotes a signal of the same "shape" but with a time delay of $a$

i.e. It doesn't start arriving until time $t=a$

<u>Ex 1:</u> Find $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2}\right\} \overset{\text{By the Theorem}}{=} u(t-3)f(t-3) \quad \text{with } F(s) = \frac{1}{s^2}$$

By the Table

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = \frac{1}{2}t^2$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = u(t-3)f(t-3) = u(t-3)\cdot\left(\frac{1}{2}(t-3)^2\right)$$

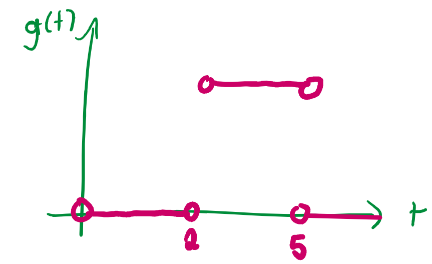
Remember

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases} \Rightarrow u(t-3) = \begin{cases} 0 & \text{if } t < 3 \\ 1 & \text{if } t \geq 3 \end{cases}$$

So $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = \begin{cases} \frac{1}{2}(t-3)^2 & \text{when } t \geq 3 \\ 0 & \text{when } t < 3 \end{cases}$

Ex 2: Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 0, & 0<t<2 \\ 3, & 2<t<5 \\ 0, & t>5 \end{cases}$

Note the graph of $g(t)$ is




We want to rewrite the function so it has the same form of the step function.

$$g(t) = \begin{cases} 0, & 0<t<2 \\ 3, & 2<t<5 \\ 0, & t>5 \end{cases} \quad \text{is same as} \quad g(t) = \begin{cases} 0-0, & 0<t<2 \\ 3-0, & 2<t<5 \\ 3-3, & t>5 \end{cases}$$

So with this new rewritten form of $f(t)$ we can split the function into 2 step functions

$$g(t) = \begin{cases} 0 & 0<t<2 \\ 3 & t>2 \end{cases} - \begin{cases} 0 & 0<t<5 \\ 3 & t>5 \end{cases}$$

Check: If $0<t<2$, $g(t) = 0-0 = 0$ ✓
If $2<t<5$, $g(t) = 3-0 = 3$ ✓
If $t>5$, $g(t) = 3-3 = 0$ ✓

So what did we do? We wrote $g(t)$ as a linear combination of unit step functions

$$g(t) = 3u_2(t) - 3u_5(t)$$

where $u_2(t) = \begin{cases} 0, & 0<t<2 \\ 1, & t>2 \end{cases}$

$u_5(t) = \begin{cases} 0, & 0<t<5 \\ 1, & t>5 \end{cases}$

Remember
$u_a(t) = u(t-a)$.

So $\mathcal{L}\{g(t)\} = 3\mathcal{L}\{u_2(t)\} - 3\mathcal{L}\{u_5(t)\}$

So $\mathcal{L}\{g(t)\} = 3\mathcal{L}\{u_2(t)\} - 3\mathcal{L}\{u_5(t)\}$

$$= 3\mathcal{L}\{u(t-2)\} - 3\mathcal{L}\{u(t-5)\}$$

$$= 3\mathcal{L}\{\underbrace{u(t-2)\cdot 1}_{\substack{\text{Note } f_2(t-2)=1 \\ \Rightarrow f_2(t)=1}}\} - 3\mathcal{L}\{\underbrace{u(t-5)\cdot 1}_{\substack{\text{Note } f_5(t-5)=1 \\ f_5(t)=1}}\}$$

By the Theorem,

$$\mathcal{L}\{g(t)\} = 3e^{-2s}\mathcal{L}\{1\} - 3e^{-5s}\mathcal{L}\{1\}$$

**Remember $\mathcal{L}\{1\} = \frac{1}{s}$**

$$= 3e^{-2s}\left(\frac{1}{s}\right) - 3e^{-5s}\left(\frac{1}{s}\right)$$

$$= \frac{3}{s}\left[e^{-2s} - e^{-5s}\right]$$

Ex 3: Find the Laplace Transform of $g(t) = \begin{cases} 0, & 0<t<5 \\ t-3, & t>5 \end{cases}$

Note we have the right format for the unit step function $u_5(t)$, namely the bounds are correct, we have 2 pieces that are continuous on their respective intervals, and the top piece is 0.

Note we need to rewrite $t-3$ in terms of $t-5$ if we wish to use the Theorem where

$$g(t) = u(t-5)\, f(t-5)$$

So $t-3 = \mathbf{1}(t-5)+\mathbf{2}$.

Hence $g(t) = \begin{cases} 0 & 0<t<5 \\ (t-5)+2 & t>5 \end{cases}$

where $f(t-5) = (t-5)+2 \Rightarrow f(t) = t+2 \quad t>0$

So $\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-5)f(t-5)\} = e^{-5s}\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{t+2\} = \mathcal{L}\{t\} + 2\mathcal{L}\{1\}$$

$$\mathcal{L}\{t+2\} = \mathcal{L}\{t\} + 2\mathcal{L}\{1\}$$
$$= \frac{1}{2}s^2 + \frac{2}{s}$$

$$= e^{-5s}\left(\frac{s^2}{2} + \frac{2}{s}\right)$$

Ex 4: Find the Laplace Transform of $g(t) = \begin{cases} 0, & 0<t<\pi/2 \\ \sin(t), & t>\pi/2 \end{cases}$

Looking at the intervals of $g(t)$, we want to use the $u_{\pi/2}(t)$ unit step function

$$u_{\pi/2}(t) = u\left(t-\frac{\pi}{2}\right) = \begin{cases} 0, & 0<t<\pi/2 \\ 1, & t>\pi/2 \end{cases}$$

We want to rewrite $g(t) = u(t-\pi/2)\, f(t-\pi/2)$.

$$\begin{cases} 0, & 0<t<\pi/2 \\ \sin(t), & t>\pi/2 \end{cases} = \left(\begin{cases} 0, & 0<t<\pi/2 \\ 1, & t>\pi/2 \end{cases}\right)\left(\begin{cases} f_{*1}(t-\pi/2), & 0<t<\pi/2 \\ f_{*2}(t-\pi/2), & t>\pi/2 \end{cases}\right)$$

Well if $0 = 0 \cdot f_{*1}(t-\pi/2)$ we can say $f_{*1}(t-\pi/2)$ can equal $0$, when $0<t<\pi/2$

Now $\sin(t) = 1 \cdot f_{*2}\left(t-\frac{\pi}{2}\right)$

But I want $f_{*2}(t)$ Not $f_{*2}(t-\pi/2)$. So can I rewrite $\sin(t)$ as some with $t-\pi/2$? Yes

**Identity: $\sin(t) = \cos\left(t-\frac{\pi}{2}\right)$**

So $f(t-\pi/2) = \begin{cases} 0 & 0<t<\pi/2 \\ \cos(x-\pi/2) & t>\pi/2 \end{cases}$ which also equals $u\left(t-\frac{\pi}{2}\right) f\left(t-\frac{\pi}{2}\right)$

Hence $f(t) = \cos(t)$.

By the theorem,

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-\pi/2)f(t-\pi/2)\} = e^{-\frac{\pi}{2}s}F(s) = e^{-\frac{\pi}{2}s}\overbrace{\mathcal{L}^{-1}\{\cos(t)\}}^{\text{use table}}$$
$$= e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$$

$$= e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$$

Ex 5: Find the Laplace Transform of $g(t) = \begin{cases} \sin(2t), & 0<t<2\pi \\ 0, & t>2\pi \end{cases}$

Note there is alot of rewriting we need to do to get $u_{2\pi}(t)\, f(t-2\pi)$

Let $w(t-2\pi) = 1-u(t-2\pi) = \begin{cases} 1-0 & \text{if } 0<t<2\pi \\ 1-1 & \text{if } t>2\pi \end{cases} = \begin{cases} 1 & \text{if } 0<t<2\pi \\ 0 & \text{if } t>2\pi \end{cases}$

So $g(t) = \sin(2t)\left[1-u(t-2\pi)\right] = \underbrace{\sin(2t)}_{\text{We can find this w/ Table}} - \underbrace{u(t-2\pi)\sin(2t)}_{\text{We can find w/ table if we rewrite } \sin(2t) \text{ as something w/ } t-2\pi}$

$\sin(2t)$ has period $\pi$

$\Rightarrow \sin(2t) = \sin(2t-\pi)$

$= \sin(2t-2\pi)$

$= \sin(2t-3\pi)$

$= \sin(2t-4\pi) = \sin(2(t-2\pi))$

So $\mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\{u(t-2\pi)\sin(2(t-2\pi))\}$

$$= \frac{2}{s^2+4} - e^{-2\pi s}F(s) \quad \text{where } f(t) = \sin(2t)$$

$$= \frac{2}{s^2+4} - e^{-2\pi s}\mathcal{L}\{\sin(2t)\}$$

$$= \frac{2}{s^2+4} - e^{-2\pi s}\cdot\frac{2}{s^2+4} = \frac{2}{s^2+4}\left(1-e^{-2\pi s}\right)$$