Section 7.6: Impulses & Delta Functions

Friday, November 21, 2025 9:31 AM

Consider the differential equation

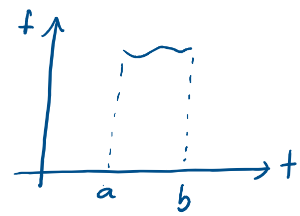
$$y'' + by' + cy = f(t)$$

We have seen that the Laplace transform is useful when the forcing term $f(t)$ is piecewise continuous. We now consider another type of forcing term, namely impulse force.

An impulse force arises when an object is dealt an instantaneous blow, like when an object is hit by a hammer.

Def: An impulse of this force, I, is

$$I = \int_a^b f(t)\,dt$$



Note based on this image we can say $f(t)$ is zero outside interval $[a,b]$. So

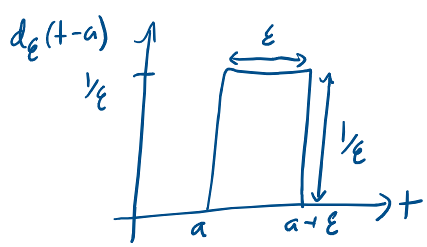
$$I = \int_{-\infty}^{\infty} f(t)\,dt$$

Sounds familar? It is like the unit step function.

Def: The unit impulse $d_{a,\varepsilon}(t)$ is defined by

$$d_{a,\varepsilon}(t) = \begin{cases} 1/\varepsilon & \text{if } a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

So the graph looks like this



$$\text{Area} = \frac{1}{\varepsilon} \cdot \varepsilon = 1 \text{ unit}$$

This means we can interpret $d_\varepsilon(t-a)$ as representing a force of magnitude $1/\varepsilon$. Note that this force does have unit impulse because of the following side note:

Side Note: First let's check for unit impulse

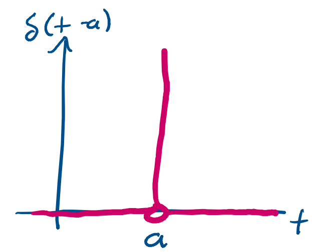
So $I = \int_{-\infty}^{\infty} d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \frac{t}{\varepsilon}\Big]_a^{a+\varepsilon} = \frac{(a+\varepsilon)-a}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = 1$

Second the idea of instantaneous force is capture when we take the limit as $\varepsilon \to 0^+$, i.e.

$$\lim_{\varepsilon \to 0^+} d_{\varepsilon,a}(t) = \lim_{\varepsilon \to 0^+} \begin{cases} 1/\varepsilon & a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases} = \underbrace{\begin{cases} \infty & t = a \\ 0 & t \ne a \end{cases}}_{\text{this has a special name.}}$$

Def: The Dirac delta function is defined by

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \ne a \end{cases} \quad \text{and} \quad \int_0^{\infty} \delta(t-a)\,dt = 1$$



Note $\delta(t-a)$ is not really a "function" but its concept is useful for the following property

$$\int_0^{\infty} g(t)\,\delta(t-a)\,dt = g(a)$$

i.e. $g(t)$ a function $\longrightarrow \int_0^{\infty} \delta(t-a)\,g(t)\,dt \longrightarrow g(a)$ a number

So what is $\mathcal{L}\{\delta(t-a)\}$?

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$

Now let's solve IVPs with forcing terms that are Dirac delta function

Ex 1: Solve IVP $\begin{cases} x'' + 4x = \delta(t-\pi) \\ x(0) = x'(0) = 0 \end{cases}$

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Like normal, take the Laplace Transform of both sides

$$\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} = \mathcal{L}\{\delta(t-\pi)\}$$

Remember

$$\mathcal{L}\{x''\} = s^2 X(s) - \underbrace{s\,x'(0)}_{0} - \underbrace{x(0)}_{0} = s^2 X(s)$$

b/c $x(0) = x'(0) = 0$

$$\mathcal{L}\{x\} = X(s)$$

So $$s^2 X(s) + 4X(s) = \underbrace{\mathcal{L}\{\delta(t-\pi)\}}_{=e^{-\pi s}}$$

$= e^{-\pi s}$ b/c $t = \pi$ there is an instantaneous force

$$X(s)\left[s^2 + 4\right] = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2 + 4}$$

Now let's take inverse Laplace Transform

$$x(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+4}\right\}$$

The Translation Theorem

$$x(t) = \mathcal{L}^{-1}\left\{e^{-as} \cdot F(s)\right\} = u(t-a)\, f(t-a)$$

So $a = \pi$ and $F(s) = \frac{1}{2} \cdot \frac{2}{s^2 + 2^2}$ $\left(f(t) = \frac{1}{2}\sin(2t)\right)$

Hence $$x(t) = u(t-\pi)\, f(t-\pi)$$

$$= u(t-\pi) \cdot \frac{1}{2}\sin(2(t-\pi))$$

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$$= \begin{cases} 0, & 0<t<\pi \\ \frac{1}{2}\sin(2t-2\pi), & t>\pi \end{cases}$$

Now let's consider an application.

Consider a physical system represented by

$$\begin{cases} ax''+bx'+cx = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

where $x(t)$ is the output or response to the input function $f(t)$.

Well let's try to solve the system like we have done with other IVPs.

Take the Laplace Transform of both sides

$$a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

Again recall

$$\mathcal{L}\{x''\} = s^2 X(s) - sx'(0) - x(0)$$
$$\mathcal{L}\{x'\} = sX(s) - x(0)$$
$$\mathcal{L}\{x\} = X(s)$$
$$\mathcal{L}\{f(t)\} = F(s)$$

And since $x(0)=x'(0)=0$

$$\mathcal{L}\{x''\} = s^2 X(s)$$
$$\mathcal{L}\{x'\} = sX(s)$$
$$\mathcal{L}\{x\} = X(s)$$
$$\mathcal{L}\{f(t)\} = F(s)$$

So $as^2 X(s) + bsX(s) + cX(s) = F(s)$

Solve for $X(s)$.

$$X(s)\left[as^2+bs+c\right] = F(s)$$

$$X(s) = \frac{F(s)}{as^2+bs+c} = F(s)\left(\frac{1}{as^2+bs+c}\right)$$

Now let $W(s) = \dfrac{1}{as^2+bs+c}$.

So $X(s) = F(s)W(s)$

by convolution property

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{F(s)W(s)\} \overset{\uparrow}{=} (f*w)(t)$$

So $\bar{X}(s) = F(s)W(s)$

by convolution property

$$x(t) = \mathcal{L}^{-1}\{\bar{X}(s)\} = \mathcal{L}^{-1}\{F(s)W(s)\} \stackrel{\uparrow}{=} (f * w)(t)$$

$$= \int_0^t w(u) f(t-u)\, du$$

Note this only happens if $x(0) = x'(0) = 0$

This result is called the Duhamel's Principle.

Additional names: $W(s) = \dfrac{1}{as^2 + bs + c}$ is the transfer function of the system

$w(t) = \mathcal{L}^{-1}\{W(s)\}$ is the weight function

Ex 2: Apply Duhamel's Principle to write an integral formula for the solution to the IVP:

$$\begin{cases} x'' + 6x' + 10x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Note $\begin{cases} x'' + 6x' + 10x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$ is the same as $\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = x'(0) = 0 \end{cases}$

where $a = 1$, $b = 6$, $c = 10$.

So $W(s) = \dfrac{1}{s^2 + 6s + 10}$

Note $s^2 + 6s + 10$ is irreducible. So let's complete the square to find its inverse Laplace transform

$$s^2 + 6s + \underline{9} + 10 - \underline{9} = (s+3)^2 + 1$$

So $W(s) = \dfrac{1}{(s+3)^2 + 1} \Rightarrow w(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+3)^2 + 1}\right\} \stackrel{\text{by the table}}{=} e^{-3t}\sin(t)$

Remember that when our IVP is of this form the solution is

$$x(t) = \mathcal{L}^{-1}\{F(s)W(s)\} = (f * w)(t)$$

$$x(t) = \mathcal{L}^{-1}\{F(s)W(s)\} = (f*w)(t)$$

$$= \int_0^t e^{-3u}\sin(u)\, f(t-u)\,du$$