Lecture 3

Section 5.2:

The Eigenvalue Method for homogeneous systems

Warm up:

Write the 1st order linear System:
$$x_1' = 2x_1 - 3x_2$$
$$x_2' = -7x_1 + x_2$$
in matrix form.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & -3 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

I. Eigenvalue Method:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

$\underline{\underline{A}}$ is a $n \times n$ constant matrix

Recall:

- 1st order linear ODE (scalar)
  $$x'(t) = \lambda x \longrightarrow x(t) = x_0 e^{\lambda t}$$

- 2nd order linear ODE
  $$ax'' + bx' + cx = 0$$
  assumed solns $x(t) = e^{rt}$
  characteristic eqn: $ar^2 + br + c = 0$
  roots r_1 and r_2
  $$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Try something Similar

Assume solns: $\underline{x}(t) = e^{\lambda t}\,\underline{v} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$

Plug into ODE:

$$\ldots = A x = A(e^{\lambda t}\underline{v}) = \cancel{e^{\lambda t}}(\underline{A}\,\underline{v})$$

Plug into ODE:

$$\underline{x}' = \lambda \cancel{e^{\lambda t}}\, \underline{v} = \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}(e^{\lambda t}\underline{v}) = \cancel{e^{\lambda t}}(\underline{\underline{A}}\,\underline{v})$$

$$\boxed{\lambda \underline{v} = \underline{\underline{A}}\,\underline{v}}$$ eigenvalue problem

identity matrix

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{x}$$

Rewrite:

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{v} = \underline{0}$$

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{\underline{I}}\,\underline{v} = \underline{0}$$

$$\underbrace{\boxed{(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}}}_{\text{solve for } \lambda \text{ and } \underline{v}}$$

NOTE: This is the vector equivalent of the characteristic equation

λ is called an eigenvalue

$\underline{v}$ is called an eigenvector

Ex: $\underline{x}' = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \underline{x}$ solns look like $e^{\lambda t}\underline{v}$

Need to solve $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$

This system has a solution when

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\lambda \underline{\underline{I}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = -\lambda(2-\lambda) - 1(3) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

characteristic eqn.

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\boxed{\begin{array}{c} \lambda = 3, -1 \\ \text{eigenvalues} \end{array}}$$

For each eigenvalue there is a corresponding

For each eigenvalue, there is a corresponding eigenvector

$\lambda_1 = 3 \longleftrightarrow \underline{v}^{(1)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

To find $\underline{v}^{(1)}$ solve $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$

$(\underline{\underline{A}} - 3\underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$

$\begin{bmatrix} 0-3 & 1 \\ 3 & 2-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

expect to have infinitely many solns

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\rightarrow$ $-3v_1 + v_2 = 0$, $\rightarrow$ $3v_1 - v_2 = 0$

one unique equation and two unknowns

$v_2 = 3v_1$ v_1 is a free variable

v_1 is a free variable, let $v_1 = 1$

$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\boxed{\underset{\text{eigenvalue}}{\lambda_1 = 3} \qquad \underset{\text{eigenvector}}{\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}}$$

$\underline{\underline{A}}\underline{v} = \lambda \underline{v}$

$\underline{\underline{A}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$2[3\underline{v}] = 2[\underline{\underline{A}}\underline{v}] = \underline{\underline{A}}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

a constant multiple of $\underline{v}$ is <u>also</u> an eigenvector

$\underline{\underline{A}}(k\underline{v}) = k(\underline{\underline{A}}\underline{v}) = k(\lambda\underline{v})$
$= \lambda(k\underline{v})$

Fundamental solution $\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{\lambda_1 t}\underline{v}^{(1)}$

Check that $\underline{x}^{(1)}$ solves $\underline{x}' = \underline{\underline{A}}\underline{x}$

Find eigenvector for $\lambda_2 = -1$ Find $\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Find eigenvector for $\lambda_2 = -1$ Find $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Solve: $(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}^{(2)} = \underline{0}$

$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 + v_2 = 0$

$3v_1 + 3v_2 = 0$

one unique eqn
2 unknowns

$v_2 = -v_1$

here v_1 is a free variable

$$\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fundamental solution $\underline{x}^{(2)} = e^{\lambda_2 t} \underline{v}^{(2)} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Use the Principle of Superposition to find the general soln:

$$\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t)$$

$$\boxed{\underline{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Stop Here

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II. Graphical Interpretation:

eigenvalues

$\lambda_1 = 3$

$\lambda_2 = -1$

eigenvectors

$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Think of $T(\underline{x}) = \underline{\underline{A}}\,\underline{x}$ as transformation