Find eigenvector for $\lambda_2 = -1$ Find $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Solve: $(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}^{(2)} = \underline{0}$

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 + v_2 = 0$

$3v_1 + 3v_2 = 0$

one unique eqn
2 unknowns

$v_2 = -v_1$

here $v_1$ is a free variable

$$\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fundamental solution $\underline{x}^{(2)} = e^{\lambda_2 t}\underline{v}^{(2)} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Use the Principle of Superposition to find the general soln:

$$\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t)$$

$$\boxed{\underline{x}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

## II. Graphical Interpretation:

eigenvalues

$\lambda_1 = 3$

$\lambda_2 = -1$

eigenvectors

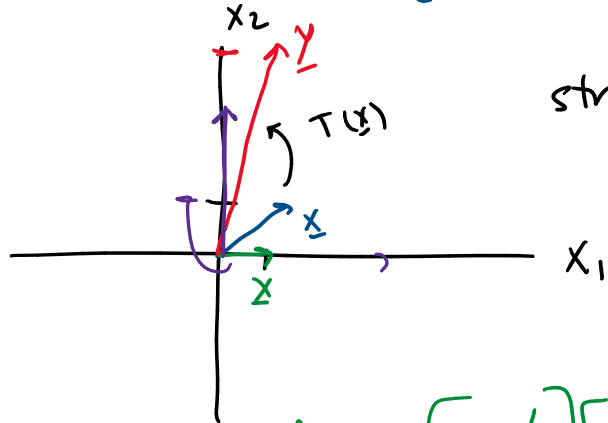
$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Think of $T(\underline{x}) = \underline{\underline{A}}\underline{x}$ as transformation

$\underline{x} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \underline{\underline{A}}\,\underline{x}$

$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$



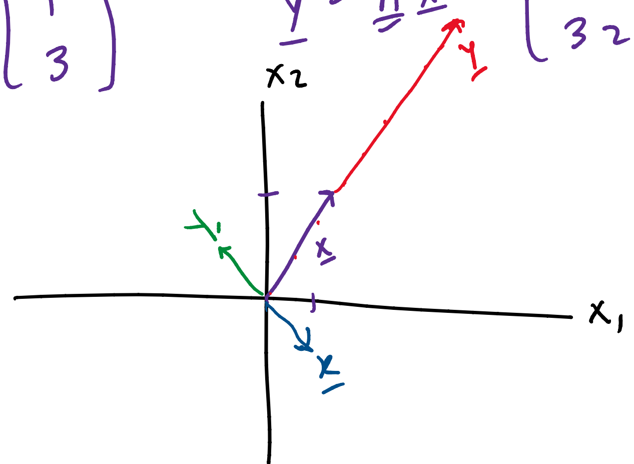

stretches and rotates the vector $\underline{x}$

$\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$T(\underline{x})$ stretches + rotates $\underline{x}$

$\underline{Q}$: what happens if $\underline{x} = \underline{v}^{(1)}$ eigenvector?

$\underline{x} = \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$




$\underline{\underline{A}}\,\underline{v}^{(1)} = 3\,\underline{v}^{(1)}$

T stretches by +3

no rotation

$\underline{x} = \underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = -1\,\underline{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Edit: $\underline{\underline{A}}(e^{\lambda_1 t}\underline{v}^{(1)}) = e^{\lambda_1 t}(\lambda_1 \underline{v}^{(1)})$

In terms of the ODE

$\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix} 1 \\ \cdots \end{bmatrix}$

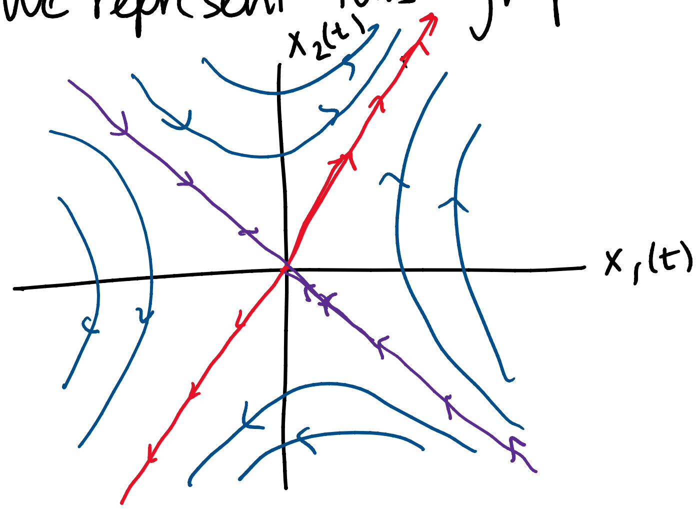
$\dfrac{d\underline{x}^{(1)}}{dt} = \underline{\underline{A}}\,\underline{x}^{(1)} = \lambda_1 \underline{x}^{(1)}$

curves grow

$\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$ $\quad\quad$ $\frac{d\underline{x}}{dt}$

solution curves grow exponentially along $\underline{v}^{(1)}$

We represent this graphically in a <u>phase portrait</u>




1. Draw eigenvectors
2. If $\lambda$ is ⊕ grow
   $\lambda$ is ⊖ decay
3. draw outside curves

$$\frac{d\underline{x}^{(2)}}{dt} = \underline{\underline{A}}\,\underline{x}^{(2)} = \lambda_2 \underline{x}^{(2)}$$

solution curves decay exponentially to origin along $\underline{v}^{(2)}$

$\boxed{\text{Saddle point}}$

$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $\quad$ plotting $x_1(t)$ vs $x_2(t)$

## III. Eigenvalue Method:

Given $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ $\quad\quad$ ($\underline{\underline{A}}$ is $n \times n$ matrix)

0. Guess $\underline{x} = e^{\lambda t}\underline{v}$, plug into ODE to obtain
   $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$

1. Find $n$ eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$
   by solving $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

2. For each $\lambda_i$ find the corresponding eigenvector $\underline{v}^{(i)}$
   solve $(\underline{\underline{A}} - \lambda_i \underline{\underline{I}})\underline{v}^{(i)} = \underline{0}$

3. General solution is:

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}^{(1)} + c_2 e^{\lambda_2 t} \underline{v}^{(2)} + \dots + c_n e^{\lambda_n t} \underline{v}^{(n)}$$

4. Plug in initial condition $\underline{x}(0) = \underline{x}_0$
solve for $c_1, \dots, c_n$

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$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$\underline{\underline{A}}$ is $2 \times 2$

$$a\lambda^2 + b\lambda + c = 0$$

roots $\lambda = a \pm bi$

Q: what happens if $\lambda$ are complex valued

## IV. Complex Eigenvalues:

Ex: $\underline{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \underline{x}$

1. eigenvalues $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$$

$$\sqrt{(1-\lambda)^2} = \sqrt{-1}$$

$$1 - \lambda = \pm i$$

$$\boxed{\lambda = 1 \pm i}$$

NOTE: complex eigenvalues always appear in conjugate pairs

2. eigenvectors:

$\boxed{\lambda_1 = 1 + i}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$$

$$\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \end{bmatrix} \rightarrow -iv_1 + v_2 = 0$$

$$v_2 = iv_1$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -i v_1 + v_2 = 0$$

$$v_2 = i v_1$$

$$-v_1 - i v_2 = 0$$

$$-v_1 = i v_2$$

$$i v_1 = (-i)(-v_1) = -\frac{v_1}{i} = v_2 \qquad \left(\frac{1}{i} = -i\right)$$

$v_1$ is a free variable, choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ i v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\boxed{\lambda_2 = 1 - i} \qquad (\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\, \underline{v}^{(2)} = \underline{0}$$

$$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

eigenvectors are conjugate pairs

Fundamental solutions:

$$\underline{x}^{(1)} = e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad \underline{x}^{(2)} = e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

one general solution

$$\underline{x}(t) = c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

WANT: Real-valued solution

Find a set of fundamental solns $\hat{\underline{x}}^{(1)}$ and $\hat{\underline{x}}^{(2)}$ that are real-valued.

Euler's formula $\quad e^{it} = \cos(t) + i \sin(t)$

$$\underline{x}^{(1)} = e^{t} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{t} \left( \cos(t) + i \sin(t) \right) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{t} \begin{bmatrix} \cos(t) + i \sin(t) \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{bmatrix}$$

$$= e^t \left\{ \underbrace{\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}}_{\underline{w}} + i \underbrace{\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}_{\underline{v}} \right\} = e^t(\underline{w} + i\underline{v})$$

Since $\underline{x}^{(2)}$ is complex conj of $\underline{x}^{(1)}$

$$\underline{x}^{(2)} = e^t(\underline{w} - i\underline{v})$$

New basis ~~for~~ fundamental solns

$$\hat{\underline{x}}^{(1)} = \frac{1}{2}(\underline{x}^{(1)} + \underline{x}^{(2)}) = \frac{1}{2}\left[e^t\{\underline{w} + \cancel{i\underline{v}}\} + e^t\{\underline{w} - \cancel{i\underline{v}}\}\right]$$

$$\boxed{\hat{\underline{x}}^{(1)} = e^t \underline{w}}$$

$$\hat{\underline{x}}^{(2)} = \frac{1}{2i}(\underline{x}^{(1)} - \underline{x}^{(2)}) = e^t \underline{v}$$

Real-valued solution

$$\boxed{\underline{x}(t) = c_1 e^t \underline{w} + c_2 e^t \underline{v}}$$

$$\boxed{= c_1 e^t \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}$$

Next time: phase portrait