Lecture 6

★Section 5.5 - Part 1

Multiple Eigenvalues

Bonus Material

Warm up:

Find the eigenvalues of the matrix
$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 \\ 7 & 2 \end{bmatrix}$$

Ans: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$
$$\begin{vmatrix} 2-\lambda & 0 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 0.7 = 0$$
$$(2-\lambda)^2 = 0$$
$$\lambda = 2$$
multiplicity $k = 2$

I. Complex Eigenvalues:

$$\underline{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x} \qquad \lambda = 1 \pm i$$

eigenvectors $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

One possible general soln:
$$\underline{x}(t) = c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

WANT: Real-valued solutions

Euler's formula: $\underline{e^{it} = \cos(t) + i\sin(t)}$

$$\underline{x}^{(1)}(t) = e^t \underbrace{e^{it}} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t \left\{ \underbrace{\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}}_{\underline{w}} + i \underbrace{\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}_{\underline{v}} \right\}$$

$$\underline{x}^{(2)}(t) = e^{t}\left\{\underline{w} - i\underline{v}\right\}$$

New basis:

$$\hat{\underline{x}}^{(1)}(t) = \frac{1}{2}\left(\underline{x}^{(1)} + \underline{x}^{(2)}\right) = e^{t}\underline{w}$$

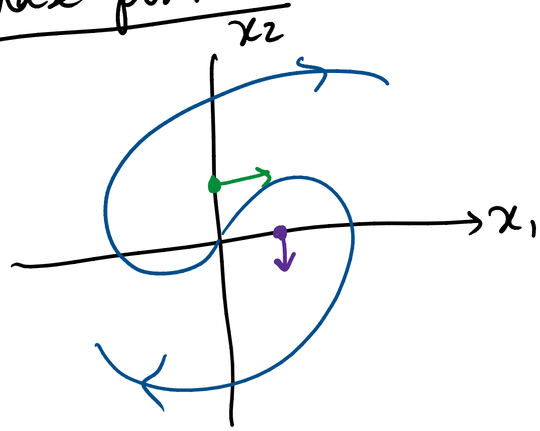
$$\hat{\underline{x}}^{(2)}(t) = \frac{1}{2i}\left(\underline{x}^{(1)} - \underline{x}^{(2)}\right) = e^{t}\underline{v}$$

Real valued solution:

$$\underline{x}(t) = C_1 e^{t}\underline{w} + C_2 e^{t}\underline{v}$$

$$\boxed{\underline{x}(t) = C_1 e^{t}\begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix} + C_2 e^{t}\begin{bmatrix}\sin(t)\\ \cos(t)\end{bmatrix}}$$

phase portrait:



$e^{t} \rightarrow$ exp growth

$\underline{w}, \underline{v} \rightarrow$ oscillate around $(0,0)$

out spiral

$\boxed{\text{spiral source}}$

(spiral sink if arrows point to origin)

Q: Clockwise (CW) or Counter clockwise (CCW)

Evaluate $\underline{u}(t)$ and $\underline{v}(t)$ @ $t=0$

$\underline{u} = \begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix}$ $\quad \underline{u}' = \begin{bmatrix}-\sin(t)\\ -\cos(t)\end{bmatrix}$ $\quad \underline{u}(0) = \begin{bmatrix}1\\ 0\end{bmatrix}$ $\quad \underline{u}'(0) = \begin{bmatrix}0\\ -1\end{bmatrix}$

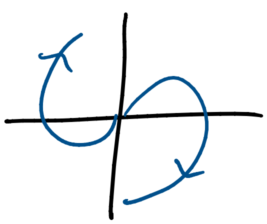
$\underline{v} = \begin{bmatrix}\sin(t)\\ \cos(t)\end{bmatrix}$ $\quad \underline{v}' = \begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix}$ $\quad \underline{v}(0) = \begin{bmatrix}0\\ 1\end{bmatrix}$ $\quad \underline{v}'(0) = \begin{bmatrix}1\\ 0\end{bmatrix}$

streamlines move in CW direction

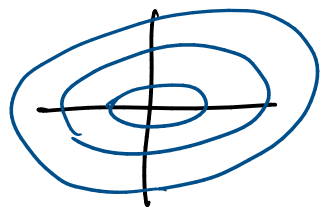
phase portraits for complex eigenvalues

$\lambda = a \pm bi$ $\qquad$ $\text{Real}(\lambda) = a$

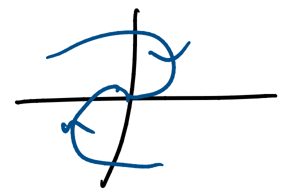
$\text{Real}(\lambda) > 0$ $\qquad$ $\text{Re}(\lambda) = 0$ $\qquad$ $\text{Real}(\lambda) < 0$



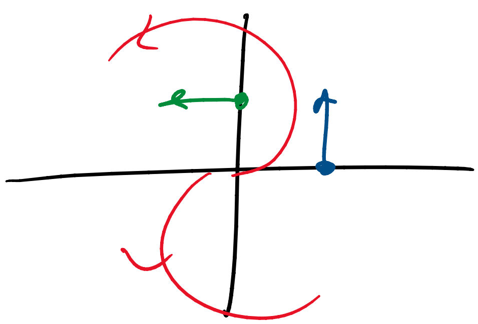
spiral source



center



spiral sink



CCW spiral

$\underline{u}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ $\underline{u}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

II Multiple Eigenvalue Solutions:

→ Lecture started Here

In chap 3 (MA 266) we solved:

$$y'' - 6y' + 9y = 0$$

convert to a system of ODE

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}$$

these 2 ODEs are equivalent

Compare

	1D	2D
eqn	$y'' - 6y' + 9y = 0$	$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}$
char eqn.	$r^2 - 6r + 9 = 0$	$\det(\underline{A} - \lambda \underline{\underline{I}}) = 0$ $\begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = 0$ $\lambda^2 - 6\lambda + 9 = 0$
roots/ eigenvalues	$(r-3)^2 = 0$ $r = 3$ multiplicity $k=2$ (repeated root)	$(\lambda - 3)^2 = 0$ $\lambda = 3$ algebraic multiplicity $k=2$ (# of times eigenvalue is repeated)
fundamental solutions	$y_1(t) = e^{3t}$ need a 2nd linearly independent soln → multiply by t (Reduction of Order) $y_2(t) = te^{3t}$	$\lambda = 3$ $(\underline{A} - 3\underline{\underline{I}})\underline{v} = \underline{0}$ $\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-3v_1 + v_2 = 0$ v_1 is a free variable $v_2 = 3v_1$ $\underline{v} = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ choose $v_1 = 1$ only 1 eigenvector geometric multiplicity $\rho = 1$ (# of eigenvectors correspond to λ) Need to find a 2nd linearly independent vector $\underline{u}$

NOTE: When

p geometric multiplicity (# of eigenvectors) $<$ k algebraic multiplicity (# times λ is repeated)

We say that the eigenvalue λ is defective

Define the defect $d = k - p$

In our example: $p = 1 < k = 2$

so $\lambda = 3$ is defective with $d = k - p = 1$

one fundamental solution

$$\underline{x}^{(1)} = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Need to find a generalized eigenvector $\underline{u}$

In 2D: solve $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u} = \underline{v}$

NOTE: This also means $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{u} = \underline{0}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{u} = (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underbrace{(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u}}_{\underline{v}} = (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$$

b/c $\underline{v}$ is an eigenvector

Solve $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u} = \underline{v}$

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$-3u_1 + u_2 = 1$

$u_2 = 1 - 3u_1$

u_1 is a free variable

choose $u_1 = 0$

$$\underline{u} = \begin{bmatrix} u_1 \\ 1 - 3u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

NOTE: $\underline{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ these are linearly independent

second fundamental solution

$$\underline{x}^{(2)} = e^{3t}(t\underline{v} + \underline{u}) = e^{3t}\left\{ t\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= e^{3t}\begin{bmatrix} t \\ 3t+1 \end{bmatrix}$$

General solution:

$$\underline{x}(t) = c_1 \underline{x}^{(1)} + c_2 \underline{x}^{(2)}$$

$$\boxed{= c_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} t \\ 3t+1 \end{bmatrix}}$$

$$\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Compare to the 1D solution

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}$$

In 2D

$$x_1 = y(t) = c_1 e^{3t} + c_2 t e^{3t}$$

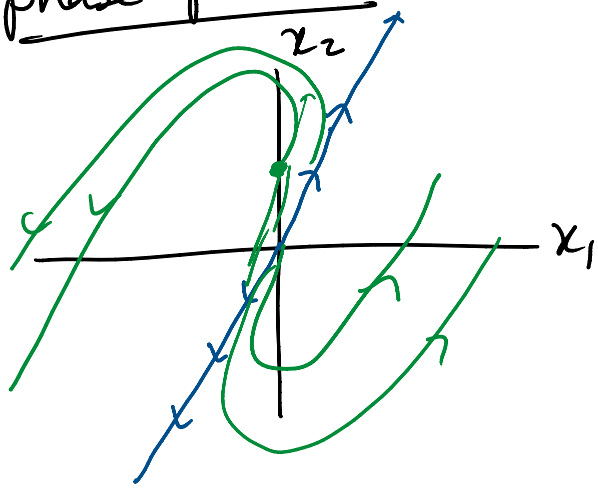
$$x_2 = y' = c_1 e^{3t} + c_2 (3t+1) e^{3t}$$

these solutions are equivalent.

Can derive $\underline{x}^{(2)}$ using Reduction of Order

$$x(t) = c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}t\\3t+1\end{bmatrix}$$

phase portrait



1. Draw eigenvector $\underline{v} = \begin{bmatrix}1\\3\end{bmatrix}$
2. $\lambda = 3$ is ⊕ so arrows point out
3. As $t \to \infty$ $e^{3t} t \begin{bmatrix}1\\3\end{bmatrix}$ dominates

   $\underline{x}$ goes parallel to $\begin{bmatrix}1\\3\end{bmatrix}$

improper nodal source

4. Let $c_1 = 0$, $c_2 = 1$

$$\boxed{\underline{x}(t) = e^{3t}\begin{bmatrix}t\\3t+1\end{bmatrix}}$$ plug into a graph

$$\underline{x}' = 3e^{3t}\begin{bmatrix}t\\3t+1\end{bmatrix} + e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$$

$$\underline{x}'(0) = 3\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix} \text{ @ } t=0 \qquad \underline{x}(0) = \begin{bmatrix}0\\1\end{bmatrix}$$

goes parallel to $\begin{bmatrix}1\\3\end{bmatrix}$

III. Complete Eigenvalues:

Not every repeated eigenvalue is defective

$$\underset{\substack{\text{geometric multiplicity}\\(\#\text{ of eigenvectors})}}{p} = \underset{\substack{\text{algebraic multiplicity}\\(\#\text{ times } \lambda \text{ is repeated})}}{k}$$

Then λ is called <u>complete</u>

Ex in 2D

$$\underline{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2 = 0$$

$\lambda = 2$

alg. mult $k = 2$

$\boxed{\lambda = 2}$ $(\underline{A} - 2\underline{I})\underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

0 equations
2 unknowns
v_1 and v_2 are free vars

want 2 linearly independent eigenvectors

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

geometric multiplicity $p = 2$

$\lambda = 2$ is <u>complete</u>!

Fundamental solutions

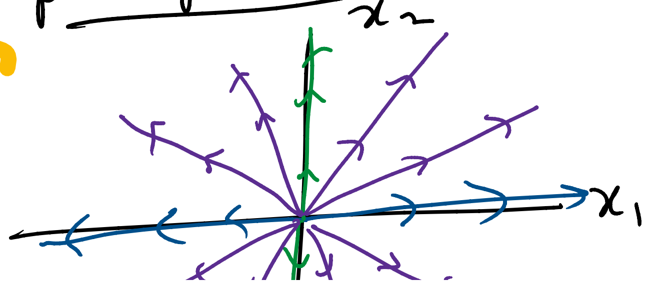
$$\underline{x}^{(1)} = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{x}^{(2)} = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

General solution

$$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Bonus

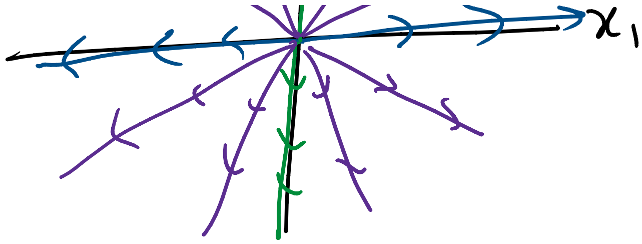
phase portrait:



1. Draw eigenvectors

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. $\lambda = 2$ is $+$
arrows point out



proper nodal source

2. $\lambda = 2$ is $+$
   arrows point out

3. as $t \to \infty$
   both $\underline{x}^{(1)}$ and $\underline{x}^{(2)}$ have e^{2t}
   grow at same rate

makes sense b/c $\underline{\underline{A}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\underline{\underline{I}}$

$\underline{x}' = \underline{\underline{A}}\,\underline{x} = 2\underline{\underline{I}}\,\underline{x} = 2\underline{x}$

every vector $\underline{x}$ is an eigenvector!

Next class (in 3D) multiple eigenvectors