

★ Section 5.5 - Part 2

Multiple EigenvaluesWarm up:

Find the eigenvalues of the matrix

$$\underline{A} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underline{\text{Ans:}} \det(\underline{A} - \lambda \underline{I}) = 0 = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} \quad \begin{array}{l} \text{expansion} \\ \text{by} \\ \text{minors} \end{array}$$

$$= (3-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - (1) \begin{vmatrix} 0 & 1 \\ 0 & 3-\lambda \end{vmatrix} + (2) \begin{vmatrix} 0 & 3-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (3-\lambda)[(3-\lambda)^2 - 0] = (3-\lambda)^3 = 0$$

$$\lambda = 3$$

algebraic mult $k = 3$ I. Multiple Eigenvalues:

p
geometric
multiplicity
(# of eigenvectors)
 $\downarrow$

k
algebraic
multiplicity
(# of times λ
is repeated)

If $p < k$, then λ is defective

... (one or more) generalized

Need to find (one or more) generalized eigenvectors

Ex: $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \lambda = 3$
 $k = 3$

Look for eigenvectors:

$$(A - 3I)\underline{v} = \underline{0}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} &\rightarrow v_2 + 2v_3 = 0 \\ &\quad v_3 = 0 \rightarrow v_2 = 0 \\ &\text{no constraints on } v_1 \\ &\quad v_1 \text{ is a free variable} \end{aligned}$$

$$\underline{v} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{geometric mult } p = 1$$

Here $p = 1 < k = 3$

so $\lambda = 3$ is defective

$$\text{defect } d = k - p = 3 - 1 = 2$$

$d = 2$ Need to find 2 generalized eigenvectors

NOTE: To find a general solution, need $n = 3$ linearly independent solutions

To find more than one generalized eigenvector,

To find more than one generalized eigenvector,
we need to calculate a chain of generalized
eigenvectors

NOTE: In 2D, our defect is at most $d=1$
so we can use $(\underline{A} - \lambda \underline{I}) \underline{u} = \underline{v}$
to calculate $\underline{u}$

In 3D, we need to do something different

II. Chain of Gen. EV.

1. Calculate $(\underline{A} - \lambda \underline{I})^2$ and $(\underline{A} - \lambda \underline{I})^3$
(Need $(\underline{A} - \lambda \underline{I})^{d+1}$)

$$(\underline{A} - 3\underline{I})^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\underline{A} - 3\underline{I})^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Find a nonzero vector $\underline{u}^{(3)}$ such that
 $(\underline{A} - 3\underline{I})^3 \underline{u}^{(3)} = \underline{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3 free variables
want $\underline{u}^{(3)} \neq \underline{0}$
want $\underline{u}^{(3)}$ and $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
to be linearly indep.

Choose

could choose

Choose $\underline{u}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (could choose $\begin{bmatrix} 200 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 100 \\ -99 \\ 223 \end{bmatrix}, \text{etc.}$)

3. Calculate the chain of generalized eigenvectors

$$\underline{u}^{(2)} = (\underline{A} - 3\underline{I})\underline{u}^{(3)} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \underline{u}^{(2)}$$

$$\underline{u}^{(1)} = (\underline{A} - 3\underline{I})\underline{u}^{(2)} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{u}^{(1)}$$

4. Check that $\underline{u}^{(1)}$ is a scalar multiple of the eigenvector $\underline{v}$

$$\underline{u}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

yes $\underline{u}^{(1)} = \underline{v} \rightarrow$ chain is valid

The chain of generalized ev. is:

$$\underline{u}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{u}^{(2)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{u}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

NOTE: Each vector in the chain satisfies

$$\underline{u}^{(i)} = (\underline{A} - \lambda \underline{I})\underline{u}^{(i+1)}$$

and the chain ends on the eigenvector $\underline{v}$,

and the chain ends on the eigen...

5. Write down fundamental solns:

$$\underline{x}^{(1)} = e^{\lambda t} \underline{u}^{(1)} = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b/c $\underline{u}^{(1)}$ is an eigenvector

$$\underline{x}^{(2)} = e^{\lambda t} \left\{ t \underline{u}^{(1)} + \underline{u}^{(2)} \right\}$$

same as 2D

$$= e^{3t} \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} = e^{3t} \begin{bmatrix} t+2 \\ 1 \\ 0 \end{bmatrix}$$

t multiplies the eigenvector

no t dependence on the gen. ev. $\underline{u}^{(2)}$

$$\underline{x}^{(3)} = e^{\lambda t} \left\{ \frac{t^2}{2} \underline{u}^{(1)} + t \underline{u}^{(2)} + \underline{u}^{(3)} \right\}$$

$$= e^{3t} \left\{ \frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = e^{3t} \begin{bmatrix} \frac{t^2}{2} + 2t \\ t \\ 1 \end{bmatrix}$$

factor of $\frac{1}{2}$ in front

6. General solution $\longrightarrow$ Superposition

$$\underline{x}(t) = C_1 \underline{x}^{(1)} + C_2 \underline{x}^{(2)} + C_3 \underline{x}^{(3)}$$

$$\underline{x}(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} t+2 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} \frac{t^2}{2} + 2t \\ t \\ 1 \end{bmatrix}$$

Q: How many chains

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$n \times n$ matrix

$$\lambda = 1$$

$$k = 5 \quad p = 2$$

$$\lambda = 2$$

$$k = 4 \quad p = 1$$

need a chain for $\lambda = 1$
need a chain for $\lambda = 2$ } each defective eigenvalue

chain has length $d+1$

$$\text{start } (A - \lambda I)^{d+1} \underline{u}^{(d+1)} = \underline{0}$$

$$\underline{u}^{(d)}$$

$$\vdots$$

$$\underline{u}^{(1)}$$

Set of eigenvectors + gen. ev.

$$\{ \underline{v}^{(1)}, \dots, \underline{v}^{(p)} \} \cup \{ \underline{u}^{(1)}, \underline{u}^{(2)}, \dots, \underline{u}^{(d+1)} \}$$

$$\parallel$$

$$\underline{v}$$

★ ALGORITHM: chain of generalized e.v.

let λ be a repeated eigenvalue with defect d ($d = k - p$)

0. Find p eigenvalues $\underline{v}^{(1)}, \dots, \underline{v}^{(p)}$

1. Calculate $(A - \lambda I)^{(d+1)}$

2. Find a nonzero vector $\underline{u}^{(d+1)}$ that satisfies $(A - \lambda I)^{(d+1)} \underline{u}^{(d+1)} = \underline{0}$

• make sure $\underline{u}^{(d+1)}$ is NOT a multiple of the eigenvectors $\underline{v}^{(1)}, \dots, \underline{v}^{(p)}$

• generalized ev. $\underline{u}^{(i)}$

iii

the eigen...

- want generalized ev. $\underline{u}^{(i)}$ to be linearly independent with $\underline{v}^{(i)}$ eigenvectors

3. Calculate the chain $\underline{u}^{(i)} = (\underline{A} - \lambda \underline{I}) \underline{u}^{(i+1)}$ for $i=1, \dots, d$

4. Check that $\underline{u}^{(i)}$ is a multiple of an eigenvector
* in the span $\{\underline{v}^{(1)}, \dots, \underline{v}^{(d)}\}$ $\underline{u}$

5. Fundamental solutions

$$\underline{x}^{(1)} = e^{\lambda t} \underline{u}^{(1)}$$

$$\underline{x}^{(2)} = e^{\lambda t} \left\{ t \underline{u}^{(1)} + \underline{u}^{(2)} \right\}$$

$$\underline{x}^{(i)} = e^{\lambda t} \left\{ \frac{t^i}{i!} \underline{u}^{(1)} + \frac{t^{i-1}}{(i-1)!} \underline{u}^{(2)} + \dots + t \underline{u}^{(i-1)} + \underline{u}^{(i)} \right\}$$

6. Use Superposition to get general soln

$$\underline{x}(t) = C_1 \underline{x}^{(1)} + \dots + C_k \underline{x}^{(k)}$$

$\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ are eigenvectors with λ and λ

Q: is $\underline{v} = \underline{v}^{(1)} + \underline{v}^{(2)}$ also an ev?

$$\begin{aligned} \underline{A} \underline{v} &= \underline{A} (\underline{v}^{(1)} + \underline{v}^{(2)}) = \underline{A} \underline{v}^{(1)} + \underline{A} \underline{v}^{(2)} \\ &= \lambda \underline{v}^{(1)} + \lambda \underline{v}^{(2)} = \lambda (\underline{v}^{(1)} + \underline{v}^{(2)}) \end{aligned}$$

$$\underline{v} = \underline{v}^{(1)} + \underline{v}^{(2)} = \lambda (\underline{v}^{(1)} + \underline{v}^{(2)})$$

Ex: $\underline{A} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\lambda = 5$
 alg mult $k = 3$

$$\lambda = 5$$

$$(\underline{A} - 5\underline{I})\underline{v} = \underline{0}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$
 2 free variables v_1 and v_3

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{choose } v_1 = 1, v_3 = 0$$

$$\underline{v}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{choose } v_1 = 0, v_3 = 1$$

Need $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ to be linearly independent
 geometric mult $\boxed{p = 2}$

Defect $d = k - p = 3 - 2 = 1$ $\boxed{d = 1}$

$\lambda = 5$ is defective

Calculate chain of gen. ev.

$$1. (\underline{A} - 5\underline{I})^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2. \text{ Find } \underline{u}^{(2)} \text{ st. } (\underline{A} - 5\underline{I})^2 \underline{u}^{(2)} = \underline{0}$$

POIL: Which of the following is a valid choice for $\underline{u}^{(2)}$?
 - - ✓ - - ✓

POLL: Which of the following is a valid choice?

(a) $\begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix}$ X

multiple
of $\underline{v}^{(1)}$

(b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓

(c) $\begin{bmatrix} 25 \\ 0 \\ -7 \end{bmatrix}$ X

linear comb
of $25\underline{v}^{(1)} - 7\underline{v}^{(2)}$
not linearly indep

(d) $\begin{bmatrix} 0 \\ 999 \\ 0 \end{bmatrix}$ ✓

(e) $\begin{bmatrix} 25 \\ 101 \\ -7 \end{bmatrix}$ ✓

yes
lin indep
with $\underline{v}$

Let's take $\underline{u}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

3. Calculate the chain

$$\underline{u}^{(1)} = (\underline{A} - 5\underline{I})\underline{u}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

4. Check that $\underline{u}^{(1)}$ is in the span $\{\underline{v}^{(1)}, \underline{v}^{(2)}\}$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{v}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

yes ✓
 $\underline{u}^{(1)} = \underline{v}^{(1)}$

(If $\underline{u}^{(1)}$ is NOT a multiple of an eigenvector,
go back to step 2 and choose a new
vector $\underline{u}^{(d+1)}$)

$$\text{chain } \{\underline{u}^{(1)}, \underline{u}^{(2)}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

5. Fundamental solutions

$$\underline{x}^{(1)} = e^{\lambda t} \underline{u}^{(1)} = e^{5t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(2)} = e^{\lambda t} \{t \underline{u}^{(1)} + \underline{u}^{(2)}\} = e^{5t} \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\underline{x}^{(2)} = e^{-\lambda t} \left\{ t \underline{u}^{(1)} + \underline{u}^{(2)} \right\} = e^{5t} \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\ = e^{5t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(3)} = e^{\lambda t} \underline{v}^{(2)} = e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

6. General solution

$$\underline{x}(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

★ Summary:

- If the defect $d = k - p$ is greater than one ($d \geq 2$) then calculate a chain of generalized ev.
- Want the set of all eigenvectors and generalized eigenvectors to be linearly independent.