

Lecture: 7

★ Section 5.3:

A Gallery of Solution Curves

Warm up:

Find the trace T
and determinant D
of the matrix

$$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Ans:

$$T = \text{trace}(\underline{A}) = a + d$$

$$D = \det(\underline{A}) = ad - bc$$

For 2D linear systems:

$$\underline{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

the phase depends on the eigenvalues + eigenvectors
of $\underline{A}$

eigenvalues $\det(\underline{A} - \lambda \underline{I}) = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$
$$= \lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{T=\text{trace}(\underline{A})} \lambda + \underbrace{(ad-bc)}_{D=\det(\underline{A})} = 0$$

$$\boxed{\lambda^2 - T\lambda + D = 0} \text{ characteristic eqn.}$$

eigenvalues are roots of 2nd degree polynomial

Three major cases:

T. Real distinct λ

Three major cases:

I. Real distinct λ

II. Repeated λ w/ alg mult $k=2$

III. Complex-valued λ

I Real distinct λ

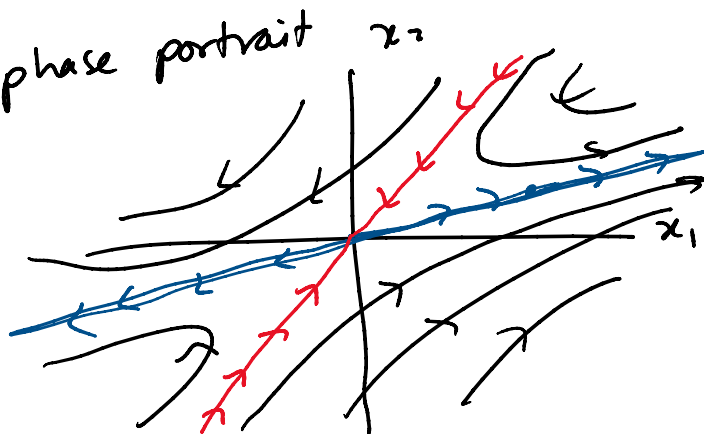
(a) λ opposite signs

Ex: $\underline{x}' = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \underline{x}$

$\lambda_1 = 2$ $\underline{v}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

phase portrait



general solution:

$$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. Draw eigenvectors

$\lambda_1 = 2 \oplus$

$\lambda_2 = -1 \ominus$

2. Draw connecting curves

Saddle point

(b) λ both negative

Ex: $\underline{x}' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \underline{x}$

$\lambda_1 = -1$ $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

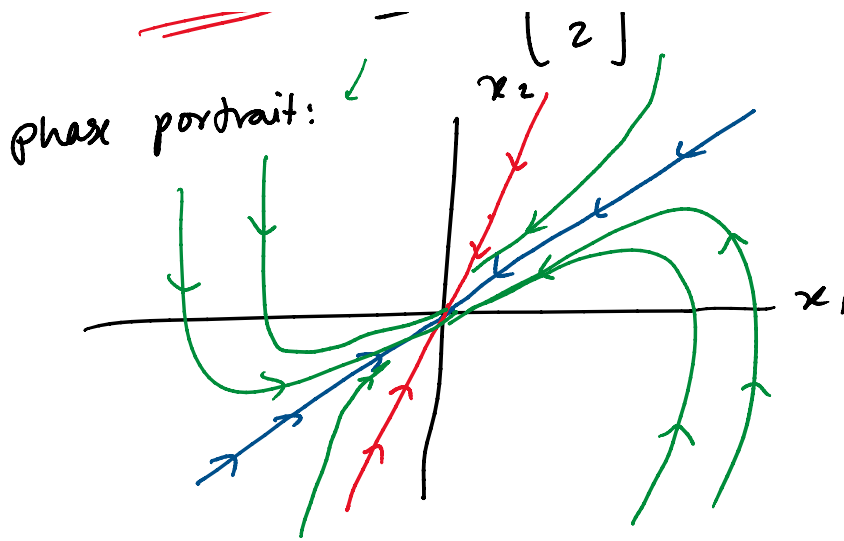
$\lambda_2 = -2$ $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

x_1 ✓ x_2 ✓

general solution

$$\underline{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1. Draw eigenvectors



1. Draw eigenvectors

$$\lambda_1 = -1 \ominus$$

$$\lambda_2 = -2 \ominus$$

2. limit as $t \rightarrow \infty$
 e^{-t} "dominates"
 $\underline{x} \rightarrow \underline{0}$ along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. limit as $t \rightarrow -\infty$
 e^{-2t} "dominates"
 $\underline{x} \rightarrow$ parallel $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

improper nodal sink

- multiple solns approach origin
- tangent to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) λ both positive

$$\underline{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

phase portrait same as (b)
 arrows point outward

improper nodal source

(d) one zero eigenvalue

Ex: $\underline{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \underline{x}$

general solution

$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

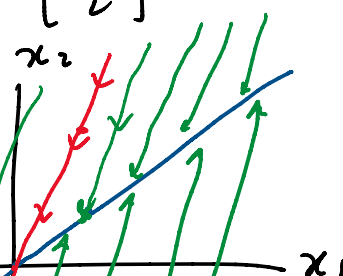
$$\lambda_1 = 0$$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1$$

$$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

phase portrait

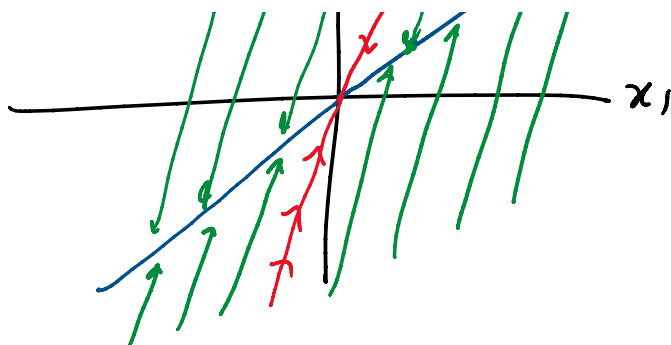


1. Draw eigenvectors

$$\lambda_1 = 0$$

$$\underline{x}^{(1)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



parallel lines

$$\underline{\lambda} = -1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{x}' = \underline{A}\underline{x}'' = 0 \underline{x}'' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 \ominus$$

2. limit as $t \rightarrow \infty$

$$\underline{x} \rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. limit as $t \rightarrow -\infty$

$$\underline{x} \rightarrow \text{parallel to } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

NOTE: $\lambda_2 < 0 \rightarrow$ stop on $\underline{v}^{(1)}$
 $\lambda_2 > 0 \rightarrow$ emanate from $\underline{v}^{(1)}$
 as $t \rightarrow \infty$, diverge

II. Real repeated λ :

(a) If λ is complete

geometric mult
 p

=

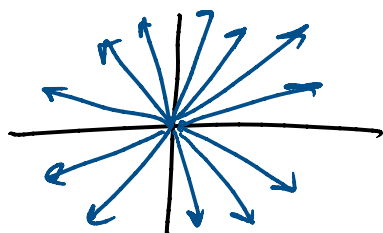
algebraic mult $k=2$

of eigenvectors = 2

2 linearly independent eigenvectors $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$

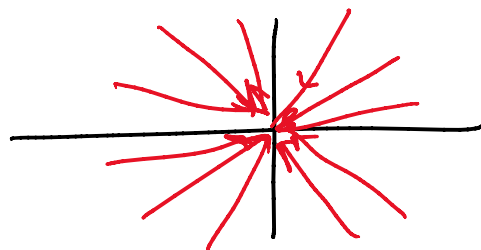
General solution $\underline{x}(t) = c_1 e^{\lambda t} \underline{v}^{(1)} + c_2 e^{\lambda t} \underline{v}^{(2)}$

if $\lambda > 0$



proper nodal source

if $\lambda < 0$



proper nodal sink

proper node source

$$\underline{x'} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$$

$$\underline{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$$

proper nodal sink

$$\underline{x}' = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \underline{x}$$

(b) If λ is defective
geometric mult
 $p = 1$

< algebraic mult $k=2$

of eigenvectors = 1

Ex: $\underline{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{x}$

$$\lambda = 3 \quad \text{with} \quad k = 2$$

$$\boxed{\lambda=3} \quad (\underline{A} - 3\underline{I}) \underline{v} = \underline{0}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} v_1 + v_2 &= 0 \\ v_2 &= -v_1 \end{aligned}$$

$$V_1 + V_2 = 0$$
$$V_2 = -V_1$$

one free variable $\rightarrow$ one eigenvector $\rightarrow p=1$

$$\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$ is defective with defect $d = 1$

Generalized eigenvector 4

$$(\underline{A} - 3\underline{I}) \underline{u} = \underline{v}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow$$

$$u_1 + u_2 = 1$$

$$u_2 = 1 - u_1$$

choose $u_1 = 0$

$$\underline{u} = \begin{bmatrix} u_1 \\ 1 - u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

General solution:

al solution:

$$\underline{x}(t) = c_1 e^{\lambda t} \underline{v} + c_2 e^{\lambda t} \left\{ t \underline{v} + \underline{u} \right\}$$

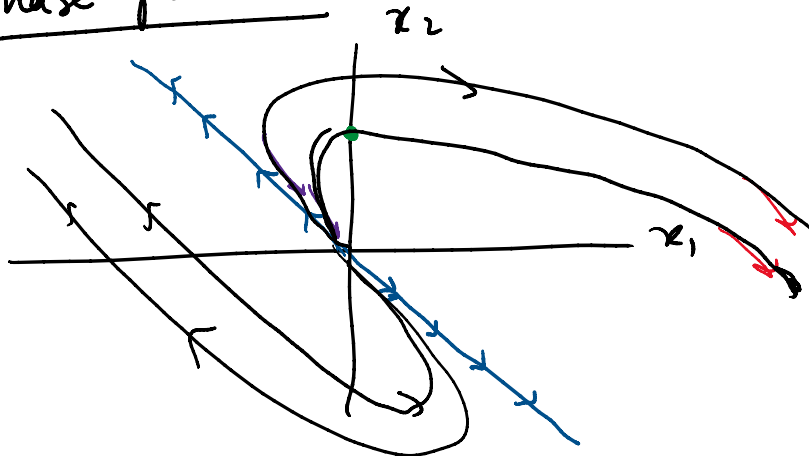
$3t \quad \quad \quad 3t \quad \quad \quad 3t \quad \quad \quad 3t$

345, 57, 507

$$\underline{x}(t) = c_1 e^{3t} \underline{v}_1 + c_2 e^{3t} \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

phase portrait:



$$\lim_{t \rightarrow -\infty} t e^{3t} = \lim_{t \rightarrow -\infty} \frac{t}{e^{-3t}}$$

L'Hôpital's rule

$$= \lim_{t \rightarrow -\infty} \frac{1}{-3e^{-3t}} = 0^-$$

improper nodal source

1. Draw eigenvector $\underline{v}$
 $\lambda = 3 \oplus$

2. let $c_1 = 0, c_2 = 1$

$$\hat{\underline{x}} = t e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

@ $t=0$ $\hat{\underline{x}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. as $t \rightarrow +\infty$
 $t e^{3t}$ dominates
 $\underline{x}$ parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. as $t \rightarrow -\infty$
 $e^{3t} \rightarrow 0$
 $t e^{3t} \rightarrow 0$
 $\underline{x} \rightarrow \underline{0}$ parallel to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

another way to look at this:

$$\hat{\underline{x}} = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

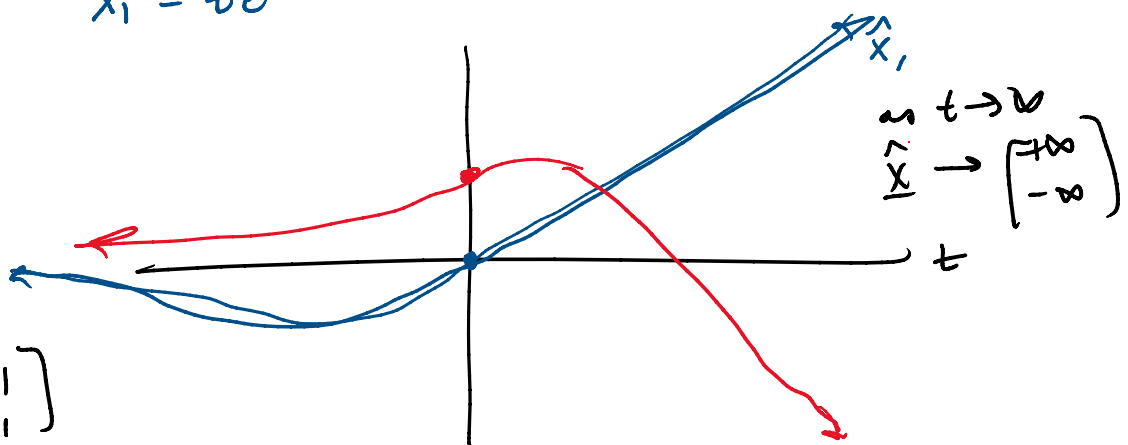
$$\hat{x}_1 = t e^{3t}$$

$$\hat{x}_2 = -t e^{3t} + e^{3t} = (1-t) e^{3t}$$

as $t \rightarrow \infty$

$$\hat{\underline{x}} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

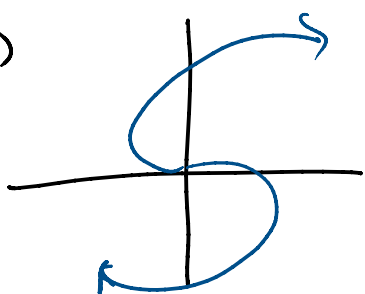
along $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



III. Complex λ : $\lambda = a \pm bi$

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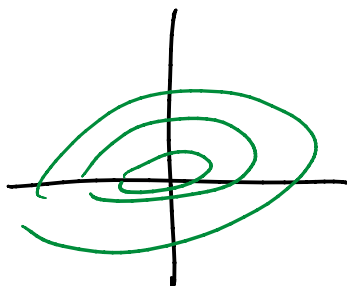
(a)



$$a > 0$$

Spiral source

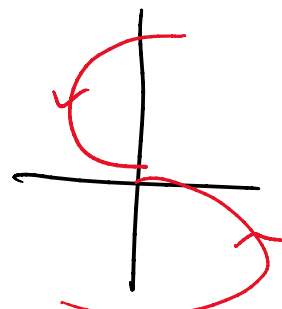
(b)



$$a = 0$$

Center

(c)



$$a < 0$$

Spiral sink

look at general solution to determine direction of rotation

$$\lambda = a \pm ib \quad \underline{x}(t) = e^{at} \left\{ c_1 \underline{w} + c_2 \underline{v} \right\}$$

evaluate $\begin{Bmatrix} \underline{w}(0) & \underline{w}'(0) \\ \underline{v}(0) & \underline{v}'(0) \end{Bmatrix}$

$$\underline{\text{Ex:}} \quad \underline{x}' = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix} \underline{x}$$

$$\lambda = \pm 10i \rightarrow \text{center}$$

$$\lambda = 10i \quad \underline{v}^{(1)} = \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$$

fund. soln:

$$\underline{x}^{(1)} = e^{10it} \begin{bmatrix} 3+5i \\ 4 \end{bmatrix} = (\cos(10t) + i \sin(10t)) \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos(10t) + 3i\sin(10t) + 5i\cos(10t) - 5\sin(10t) \\ 4\cos(10t) + 4i\sin(10t) \end{bmatrix}$$

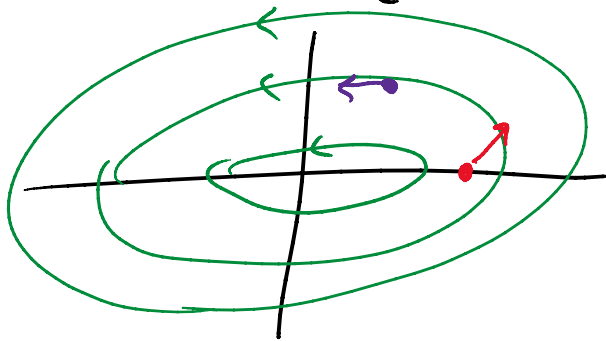
$$= \begin{bmatrix} 3\cos(10t) - 5\sin(10t) \\ 4\cos(10t) \end{bmatrix} + i \begin{bmatrix} 3\sin(10t) + 5\cos(10t) \\ 4\sin(10t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 \cos(10t) - 5 \sin(10t) \\ 4 \cos(10t) \end{bmatrix}}_{\underline{w}} + \underbrace{\begin{bmatrix} 3 \sin(10t) + 5 \cos(10t) \\ 4 \sin(10t) \end{bmatrix}}_{\underline{v}}$$

real valued solution:

$$\underline{x}(t) = C_1 \underline{w} + C_2 \underline{v}$$

$$= C_1 \begin{bmatrix} 3 \cos(10t) - 5 \sin(10t) \\ 4 \cos(10t) \end{bmatrix} + C_2 \begin{bmatrix} 3 \sin(10t) + 5 \cos(10t) \\ 4 \sin(10t) \end{bmatrix}$$



Q: arrows CW or CCW?

$$2. \underline{w}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \underline{w}'(0) = \begin{bmatrix} -50 \\ 0 \end{bmatrix}$$

$$3. \underline{v}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad \underline{v}'(0) = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

arrows point in CCW direction,