Lecture 9

★ Sec 6.1

Stability in the Phase Plane

Warm up:

The linear system:

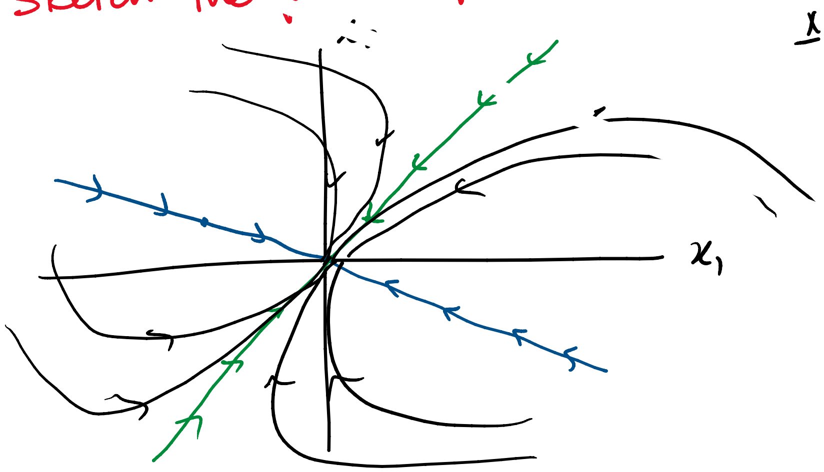
$$\underline{x}' = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix} \underline{x}$$

has eigenvalues and eigenvectors

$\lambda_1 = -5$, $\underline{v}^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_2 = -1$, $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sketch the phase portrait and identify its type.

$$\underline{x} = c_1 e^{-5t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



improper nodal sink

I. Nonlinear Systems:

A system of equations

$$(*) \begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$$

$$x' = \frac{dx}{dt}$$

is called autonomous if the RHS has no t-dependence

Def: a critical point of system $(*)$ is a point (x_*, y_*) where both

... and $G(x_*, y_*) = 0$

point (x_*, y_*) when ...

$F(x_*, y_*) = 0$ and $G(x_*, y_*) = 0$

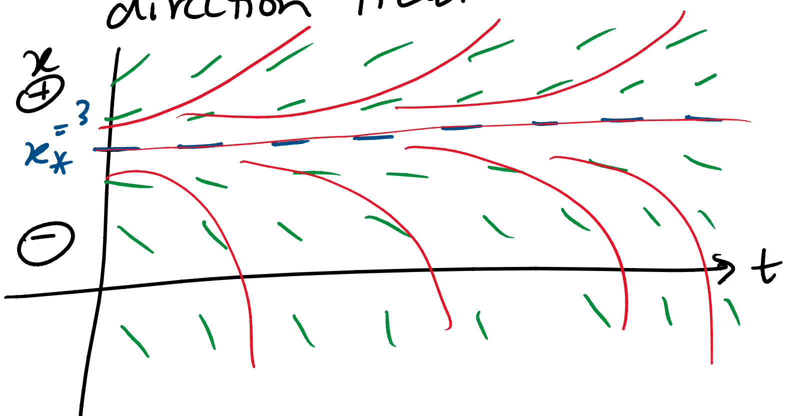
$$x'\big|_{(x_*, y_*)} = F(x_*, y_*) = 0$$
$$y'\big|_{(x_*, y_*)} = G(x_*, y_*) = 0$$

Recall: in 1D, we used critical points (c.p.) and direction fields to graphically represent solution curves

Ex: $x' = x - 3$

critical point $x_* - 3 = 0 \rightarrow x_* = 3$

direction field



@ $x_* = 3$ $x' = 0$

if $x > 3$
$x' = x - 3 > 0$ +

if $x < 3$
$x' = x - 3 < 0$ −

Sketch solution curves

if $x(t) = x_* = 3$ $x' = \frac{d}{dt}(3) = 0$
solves ODE $\rightarrow$ equilibrium solution

The solution curves diverge away from $x_* = 3$ so the critical point $x_* = 3$ is unstable

GOAL: Do same thing in 2D

1D 2D

	1D	2D
critical point	x_*	(x_*, y_*)
graph	direction field x vs t	phase portraits y vs. x think of t as a parameterization ($y(t)$ vs. $x(t)$)

Ex: $\begin{cases} x' = x - 3 \\ y' = x + 5y + 2 \end{cases}$

critical points:

$$x_* - 3 = 0$$
$$x_* = 3$$

$$x_* + 5y_* + 2 = 0$$
$$3 + 5y_* + 2 = 0$$
$$5y_* = -5$$
$$y_* = -1$$

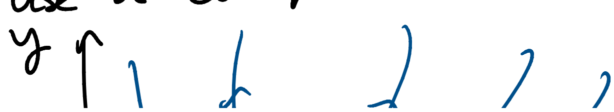
So the critical point $(3, -1)$

Then, the constant-valued function:

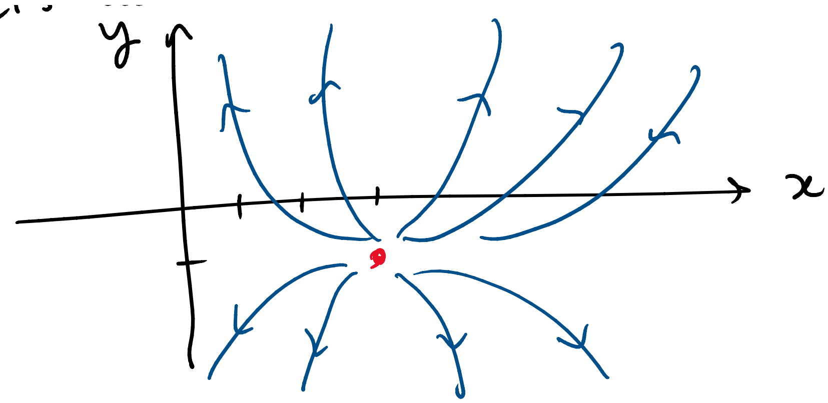
$$\begin{cases} x(t) = x_* = 3 \\ y(t) = y_* = -1 \end{cases}$$

Solve the system $(*)$. We call this an equilibrium solution

Let's use a computer to calculate phase portrait:



critical point at $(3, -1)$



critical point at $(3,-1)$

@ c.p. $x'=0$ and $y'(0)$

solution curves look like an improper nodal source shifted from $(0,0) \longrightarrow (3,-1)$

From this, since solution curves diverge away from $(3,-1)$ $\rightarrow$ we say its unstable critical point

NOTE: We have seen already all the possible behaviors of a c.p.

- (im) proper nodal source / sink
- saddle point
- spiral source / sink
- center
- parallel lines

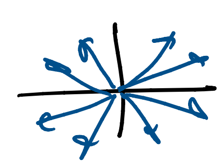
same in nonlinear systems

From the phase portrait, we can determine the stability of the c.p.

Three types:

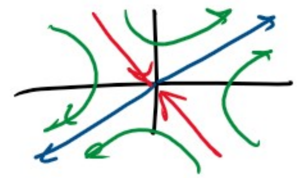
1. unstable: (some) solutions diverge away from the critical point

   Ex: proper nodal source



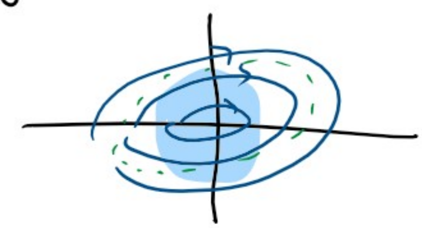
Ex: proper nodal source

Ex: saddle point



2. Stable: solutions that start "close" to the critical point stay close to the c.p.

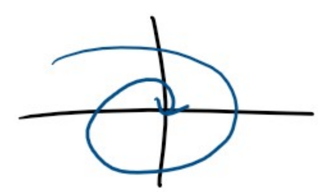
Ex: center



"don't diverge away"

3. Asymptotically stable: as $t \to \infty$, the solutions converge* to the critical point

Ex: spiral sink



*nuance: for asymptotically stable c.p. NOT EVERY solution curve will converge to c.p.

BUT

if it starts "close enough" to (x_*, y_*) the solution will converge to (x_*, y_*)

Ex: $\begin{cases} x' = 1 - y^3 \\ y' = x^2 - 4y \end{cases}$

1. Find the critical points

$1 - y_*^3 = 0$

$1 = y_*^3$

$y_* = 1$

$x_*^2 - 4y_* = 0$

$x_*^2 - 4 = 0$

$x_*^2 = 4$

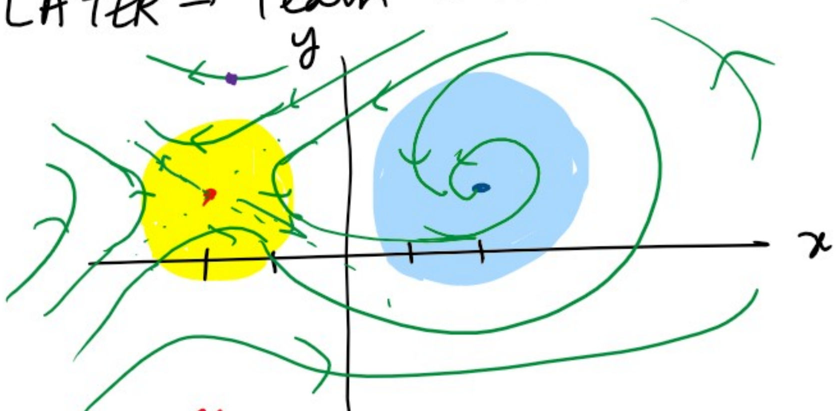
$1 = y_*$
$y_* = 1$

$x_*^2 = 4$
$x_* = \pm 2$

we have <u>two</u> critical points
$(2,1)$ and $(-2,1)$

Nonlinear systems often have multiple c.p.

LATER → learn to draw phase portraits



given the phase portrait

$(2,1)$

$(-2,1)$

saddle point

unstable

spiral sink

asymptotically stable

Look for a phase portrait w/ c.p. at $(2,-1)$ and $(2,-1)$

We can use phase portraits and critical points to analyze higher order <u>scalar</u> ODEs

<u>Ex</u>: $x'' + 2x + x^2 = 0$ scalar 2nd order nonlinear

Convert to a 2nd order system:

let $x = x$, $y = x'$

$$\begin{cases} x' = y \\ y' = x'' = -2x - x^2 \end{cases}$$

$y' = x'' = -2x - x$

1. Find the critical points

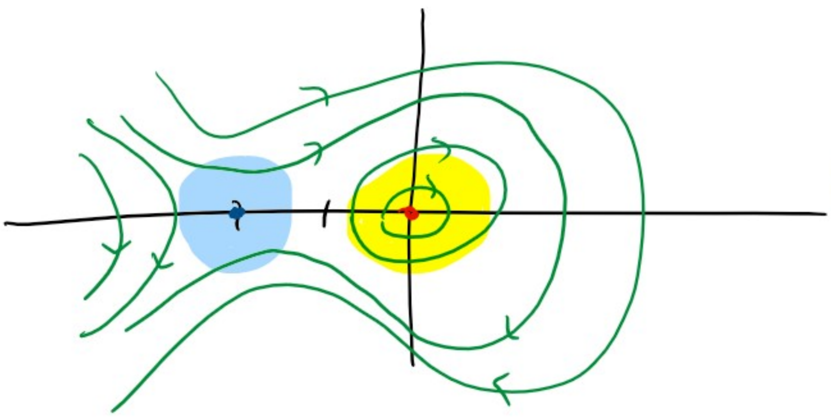
$y_* = 0$

$-2x_* - x_*^2 = 0$

$-x_*(2 + x_*) = 0$

$x_* = 0, -2$

critical points:
$(0,0)$ and $(-2,0)$

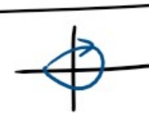


$(0,0)$ looks like a <u>center</u>
<u>stable</u>

$(-2,0)$ looks like a <u>saddle point</u>
→ <u>unstable</u>

<u>Exercise</u>: Let's sort all the possible behaviors by their stability:

<u>behaviors</u>:
- (im)proper nodal source / sink
- saddle point
- spiral source / sink
- center
- parallel lines

unstable	stable	asymptotically stable
improper nodal source	center 	(im)proper nodal sink
proper nodal source	parallel lines ($\lambda_2 < 0$)	spiral sink 

proper nodal source saddle point spiral source parallel lines ($\lambda_2>0$)	parallel lines ($\lambda_2<0$)	spiral sink

parallel lines stability:

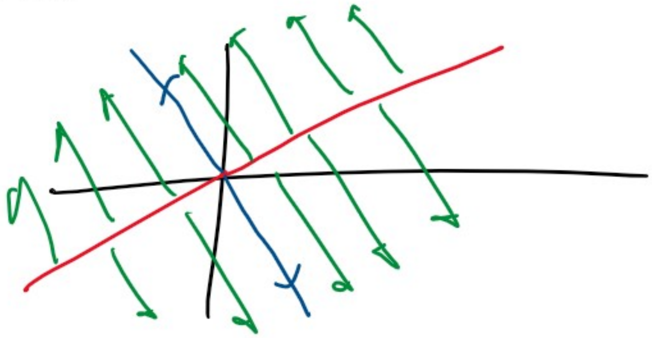
2 cases: (1) $\lambda_1=0$ $\lambda_2>0$

(2) $\lambda_1=0$ $\lambda_2<0$

case 1: $\lambda_2>0$ $+$

$\lambda_1=0$ $\underline{v}^{(1)}$

$\lambda_2>0$ $\underline{v}^{(2)}$



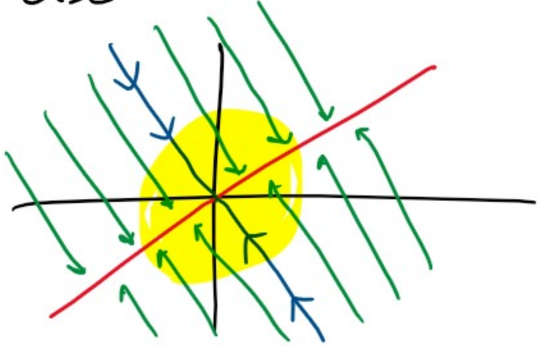
<u>unstable</u>

case 2: $\lambda_2<0$ $-$

$\lambda_1=0$ $\underline{v}^{(1)}$

$\lambda<0$ $\underline{v}^{(2)}$

$-$



<u>stable</u>