Lecture 9

☆ Section 6.2 - Part 1

Linear & Almost Linear Systems

Warm up:

Recall the Taylor Series of a function $f(x)$ at the point $x=0$ is defined by:

$$f(x) \approx f(0) + x f'(0) + \tfrac{1}{2}x^2 f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

Find the first 3 terms of the Taylor series of

$$f(x) = x^2 + 3e^x$$

Ans:

$$f(x) \approx f(0) + x f'(0) + \tfrac{1}{2}x^2 f''(0)$$

$$f(0) = \left[x^2 + 3e^x\right]\Big|_{x=0} = 0 + 3e^0 = 3$$

$$f'(0) = \left[2x + 3e^x\right]\Big|_{x=0} = 3$$

$$f''(0) = \left[2 + 3e^x\right]\Big|_{x=0} = 5$$

$$f(x) \approx \boxed{3 + 3x + \tfrac{5}{2}x^2}$$

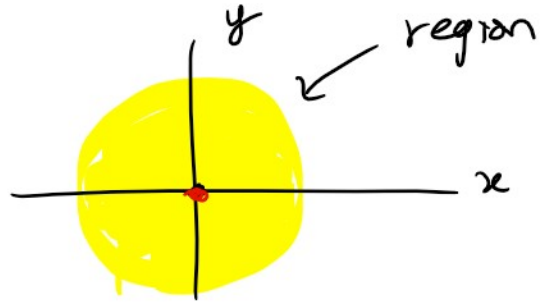
I. Stability of Linear Systems:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- the origin $(0,0)$ is always a critical point
- If $\det(\underline{A}) = ad - bc \neq 0$, then the origin

- If $\det(\underline{A}) = ad - bc \neq 0$, then the origin is an <u>isolated</u> critical point

there is a small region around (0,0) where it is the only c.p.



The characteristic eqn: $\det(\underline{A} - \lambda \underline{\underline{I}}) = 0$

$$\lambda^2 - T\lambda + D = 0$$

$$T = \text{trace}(\underline{A}) = a + d$$
$$D = \det(\underline{A})$$

The eigenvalues

$$\lambda = \frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 - 4D}$$

We can classify the critical point (0,0)

λ	Type of c.p.	stable?
real, distinct same sign	improper node	yes if both $\lambda < 0$ (asymp. stable)
real, distinct opposite sign	saddle point	NO
real, equal	proper or improper node	Yes, if $\lambda < 0$ (asymp. stable)

complex conjugate $a \pm bi$	spiral point	yes if $Re(\lambda) < 0$ (asymp. stable)
pure imaginary $\pm bi$	center	yes
$\lambda_1 = 0, \ \lambda_2 \neq 0$	parallel lines	yes if $\lambda_2 < 0$

<u>Thm:</u> (Stability of Linear Systems)

Let λ_1 and λ_2 be the eigenvalues of $\underline{x}' = \underline{A}\,\underline{x}$ where $\det(\underline{A}) \neq 0$

Then, the critical point $(0,0)$ is:

1. Asymptotically stable if $Re(\lambda) < 0$
2. Stable if $Re(\lambda) = 0$
3. unstable otherwise

By looking at the eigenvalues λ, we can assess the stability of $(0,0)$

<u>Ex:</u> $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$D = \det(\underline{A}) = (-1)2 - 6(-1) = 4$

$D \neq 0$ so $(0,0)$ is an <u>isolated</u> c.p.

$T = \text{trace}(\underline{A}) = -1 + 2 = 1$

char. eqn. $\lambda^2 - T\lambda + D = 0$

$\lambda^2 - \lambda + 4 = 0$

$$\lambda^2 - \lambda + 4 = 0$$

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{(-1)^2 - 4\cdot 1\cdot 4} = \frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$

$$Re(\lambda) = Re\left(\frac{1}{2} \pm \frac{\sqrt{15}}{2} i\right) = \frac{1}{2} > 0$$ unstable

and $(0,0)$ is a spiral source

Ex: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$D = \det(\underline{A}) = -4(2) - 1(-6) = -2$

$D \neq 0$ so $(0,0)$ is isolated

$T = \text{trace}(\underline{A}) = -4 + 2 = -2$

Char. eqn. $\lambda^2 - T\lambda + D = 0$

$\lambda^2 + 2\lambda - 2 = 0$

$$\lambda = \frac{-2}{2} \pm \frac{1}{2}\sqrt{2^2 - 4\cdot(-2)\,1}$$

$$= -1 \pm \frac{1}{2}\sqrt{12} = -1 \pm \sqrt{3}$$

$\lambda_1 = -1 + \sqrt{3} > 0$ $\qquad$ $\lambda_2 = -1 - \sqrt{3} < 0$

$Re(\lambda_1) > 0 \longrightarrow$ unstable saddle point.

II. Small Perturbations:

Q: What happens if we perturb a system by a little bit ε? (assume $\varepsilon << 1$)

still linear

<u>Ex</u>: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & -1+\varepsilon \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$D = \det(\underline{\underline{A}}) = (-1)(2) - 6(-1+\varepsilon) = -2+6-6\varepsilon =$
$= 4-6\varepsilon \quad \neq 0$ so $(0,0)$ is <u>isolated</u>

$T = \operatorname{trace}(\underline{\underline{A}}) = -1+2 = 1$

char. eqn: $\lambda^2 - T\lambda + D = 0$

$$\lambda^2 - \lambda + 4 - 6\varepsilon = 0$$

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{(-1)^2 - 4\cdot 1\cdot(4-6\varepsilon)}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 16 + 24\varepsilon}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{-15 + 24\varepsilon}$$

Q: is $-15 + 24\varepsilon > 0$

$$24\varepsilon > 15$$

$$\varepsilon > \frac{15}{24} = \frac{5}{8}$$

but $\varepsilon \ll 1$ $\Rightarrow ? \Leftarrow$

$\boxed{\varepsilon \ll 1 \text{ if } \varepsilon < \frac{1}{10}}$

NO, $-15 + 24\varepsilon < 0$

eigenvalues $\lambda = \frac{1}{2} \pm bi$

so $(0,0)$ is ^remains <u>unstable</u> and <u>spiral source</u>

remains
even after the perturbation $\varepsilon >> 1$

NOTE: In most cases, a small perturbation of $\underline{A}$ will NOT change the stability of the system

However, a perturbation can change the type of critical point (see A2)

III, Almost Linear Systems:

The Taylor series expansion of a function of two variables $f(x,y)$ at a point (x_0, y_0) is defined

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + r(x-x_0, y-y_0)$$

where $f_x(x_0,y_0) = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

$f_y(x_0,y_0) = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$

and $r(x,y)$ is called the remainder
$r(x,y) \sim O(x^2+y^2)$

If x and y are small, then $r(x,y)$ is an order of magnitude smaller

So $f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Ex: Find the Taylor series at $(0,0)$ of

$$f(x,y) = x^3 y - \sin(2x) + e^y$$

Ans:

$$f(0,0) = 0 - \sin(0) + e^0 = 1$$

$$\frac{\partial f}{\partial x}(0,0) = \left[3x^2 y - 2\cos(2x)\right]\Big|_{(0,0)} = -2$$

$$\frac{\partial f}{\partial y}(0,0) = \left[x^3 + e^y\right]\Big|_{(0,0)} = 1$$

so $f(x,y) \approx 1 + (-2)(x-0) + (1)(y-0)$

$$\approx \underbrace{1 - 2x + y}_{\text{these terms are now linear}}$$

GOAL: Use Taylor series to linearize a nonlinear system around a critical point.

Ex: $x' = F(x,y)$
$y' = G(x,y)$

Let (x_*, y_*) is a critical point of the system

By definition $F(x_*, y_*) = 0$
$G(x_*, y_*) = 0$

Now, let's Taylor expand F and G around (x_*, y_*)

$$x' = F(x,y) = \cancelto{0}{F(x_*,y_*)} + F_x(x_*,y_*)(x-x_*) + F_y(x_*,y_*)(y-y_*) + r(x,y)$$

$$y' = G(x,y) = \cancelto{0}{G(x_*,y_*)} + G_x(x_*,y_*)(x-x_*) + G_y(x_*,y_*)(y-y_*) + s(x,y)$$

b/c (x_*, y_*) is a critical point

small if $x - x_* < 1$

b/c (x_*, y_*) is a critical point

small if $x - x_* < 1$, $y - y_* < 1$

In matrix form:

$$(*)\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_*, y_*) & F_y(x_*, y_*) \\ G_x(x_*, y_*) & G_y(x_*, y_*) \end{bmatrix}}_{\underline{\underline{J}} \text{ - Jacobian matrix}} \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix} + \underbrace{\begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}}_{\text{very small near } (x_*, y_*)}$$

This is called the almost linear system

Let $x_1 = x - x_*$ and $x_2 = y - y_*$

$x_1' = x'$ $\qquad x_2' = y'$

$$(\square)\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \underline{\underline{J}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we call this the associated linear system

We can consider the almost linear system $(*)$ is a perturbation of the linear system $(\square)$

Q: What can the linear system $(\square)$ tell us about the almost linear system $(*)$?

Thm: The almost linear system $(*)$ will have the same type of critical point and the same stability as the linear system $(\square)$

$$\boxed{\text{UNLESS} \quad \lambda_1 = \lambda_2 \quad \text{or} \quad \lambda = \pm bi}$$

☆ Summary:

- a nonlinear system

$$x' = F(x,y)$$
$$y' = G(x,y)$$

has a critical point (x_*, y_*) when

$$F(x_*, y_*) = 0 = G(x_*, y_*)$$

- Use Taylor series to get the almost linear system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_*, y_*) & F_y(x_*, y_*) \\ G_x(x_*, y_*) & G_y(x_*, y_*) \end{bmatrix}}_{\underline{\underline{J}} \text{ - Jacobian matrix}} \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix} + \begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}$$

- Find the associated linear system and evaluate the <u>type</u> and <u>stability</u> of the c.p.

$$\underline{x}' = \underline{\underline{J}}\,\underline{x} \qquad \text{where} \quad \underline{x} = \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix}$$

Ex: $x' = 4x + 7y - x^2 + 3y^2$ $r(x,y)$

$y' = x - 2y - 13xy$ $s(x,y)$

Taylor series

$F(x,y) = \cancel{F(0,0)}^{\,0} + F_x(0,0)\,x + F_y(0,0)\,y + r(x,y)$

1. Claim: $(0,0)$ is a critical point

2. Calculate the Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 4-2x & 7+6y \\ 1-13y & -2-13x \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$$

Associated Linear system:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underline{\underline{J}}\,\underline{x} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

keep all the linear terms from autonomous system

$$\boxed{\begin{aligned} x' &= 4x - 7y \\ y' &= x - 2y \end{aligned}}$$